

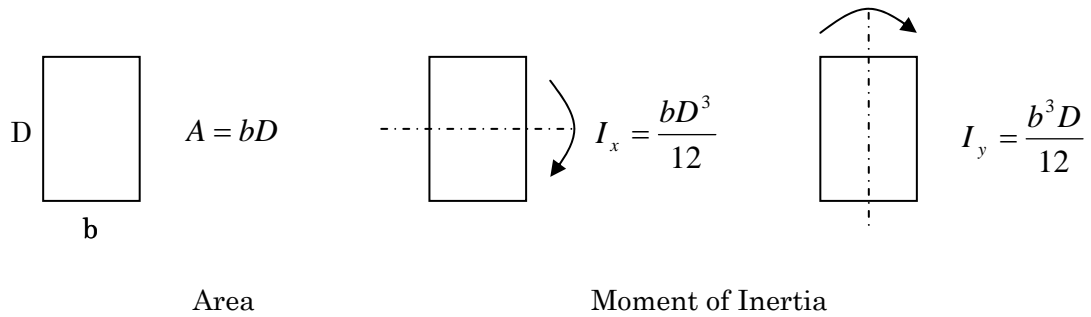
Finite Element Method 1
(Theory)

T. SAITO

Building Research Institute

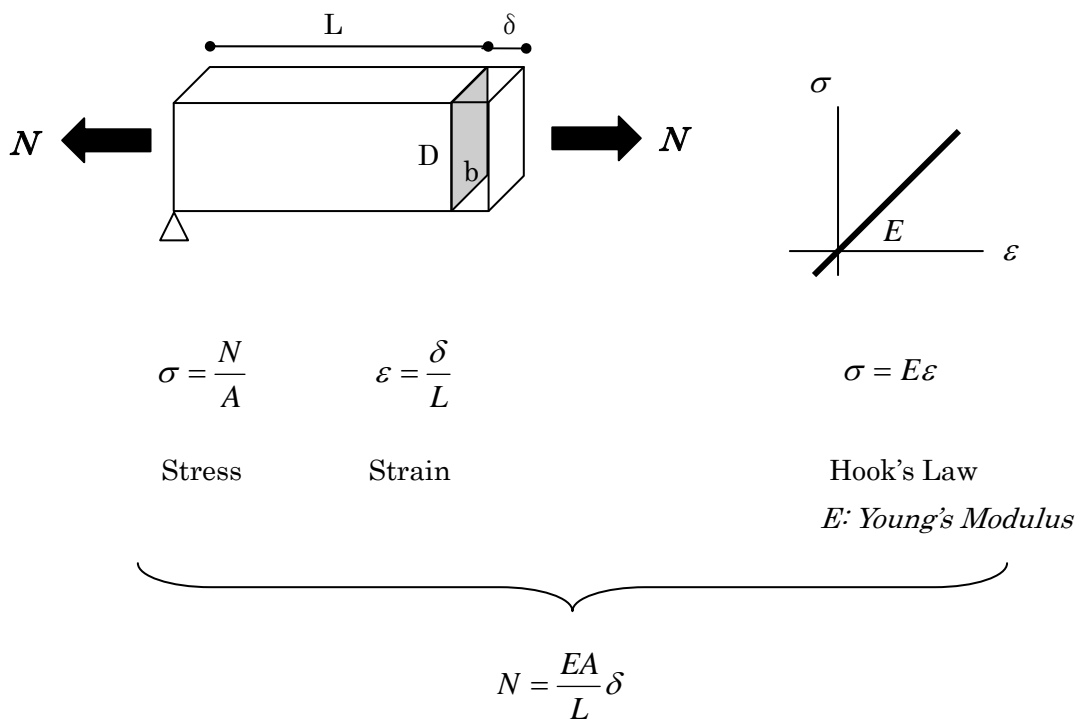
1. INTRODUCTION

1-1. Section



1-2. Stress and Strain

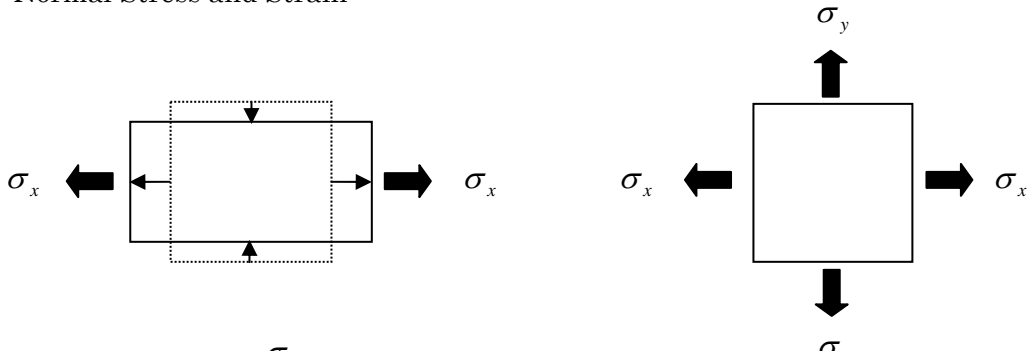
1) One-Dimensional Problem



Force – Deformation Relationship

2) Two-Dimensional Problem

Normal Stress and Strain



$$\varepsilon_x = \frac{\sigma_x}{E}$$

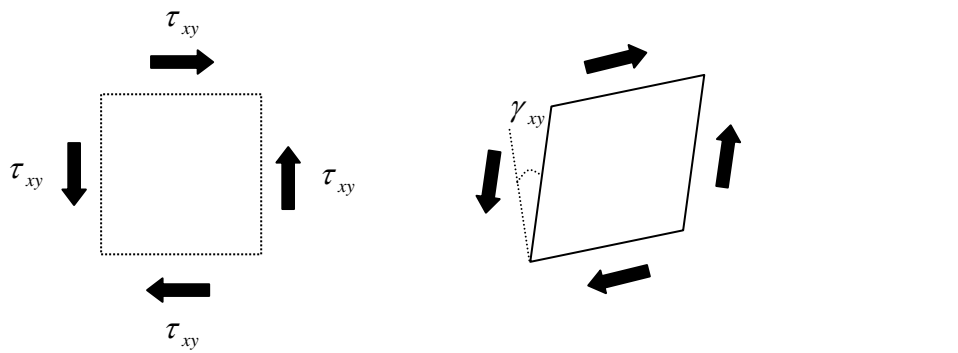
$$\varepsilon_y = -\nu \frac{\sigma_x}{E}$$

ν : Poisson Ratio

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}$$

Shear Stress and Strain

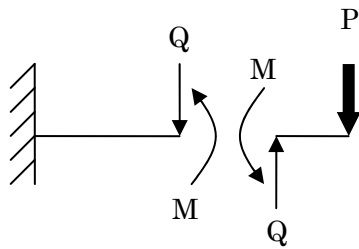
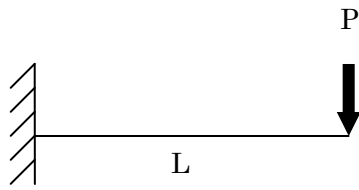


$$\tau_{xy} = G\gamma_{xy}$$

$$G = \frac{1}{2(1+\nu)} E \quad \text{: Shear Modulus}$$

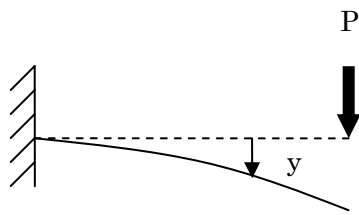
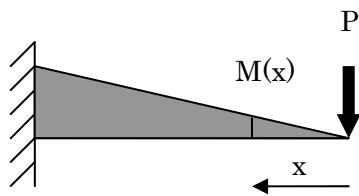
$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{bmatrix} \frac{1}{E} & -\nu \frac{1}{E} & 0 \\ -\nu \frac{1}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

1-3. Beam Theory



Q : Shear Force

M : Moment



$$\frac{d^2 y}{dx^2} = \frac{1}{EI} M(x)$$

$$\frac{dM(x)}{dx} = Q(x)$$

Example)

$$\frac{d^2 y}{dx^2} = \frac{1}{EI} M(x), \quad M(x) = Px$$

$$\rightarrow \frac{d^2 y}{dx^2} = \frac{P}{EI} x$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{P}{EI} x^2 + c_1$$

$$\rightarrow y = \frac{1}{6} \frac{P}{EI} x^3 + c_1 x + c_2$$

at $x = L$, $\frac{dy}{dx} = 0$ and $y = 0$:

Therefore,

$$c_1 = -\frac{PL^2}{2EI}, \quad c_2 = \frac{1}{3} \frac{PL^3}{EI}$$

$$y = \frac{1}{6} \frac{P}{EI} x^3 - \frac{1}{2} \frac{PL^2}{EI} x + \frac{1}{3} \frac{PL^3}{EI}$$

1.4 Properties of Reinforced Concrete Structure

Unit Weight

Concrete Type	Nominal Strength (N/mm ² = MPa)	Unit Weight (kN/m ³)
Normal Concrete	$F_c \leq 36$	24

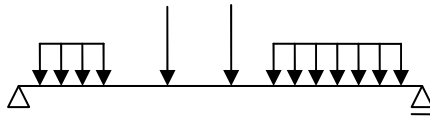
Material Parameters

	Young's Modulus (N/mm ² = MPa)	Poisson's Ratio	Thermal Expansion Coefficient (1/°C)
Steel Bar	200 000	1/4	1×10^{-5}
Concrete	22 000 ($F_c = 18$) 25 000 ($F_c = 24$) 28 000 ($F_c = 30$)	1/6	1×10^{-5}

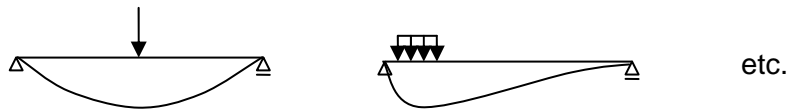
2. SIMPLE EXAMPLE FOR FEM FORMULATION

Step.1: Description of the Problem

The problem is to obtain the deformation of a simple supported beam under various load conditions.



If you change the load condition, you will get the different deformation pattern. Actually, there are infinite variations for the deformation pattern.



Step.2: Assumption of deformation function

We assume a particular function for the deformation pattern to fix the variation, such as the following function:

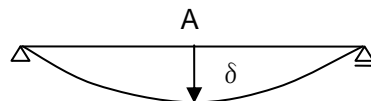
$$v(x) = a \sin\left(\frac{\pi}{L} x\right) \quad (1)$$



Step.3: Relation between nodal displacement and element deformation

From equation (1), The displacement δ at the center node A is calculated as

$$\delta = v(0.5L) = a \quad (2)$$



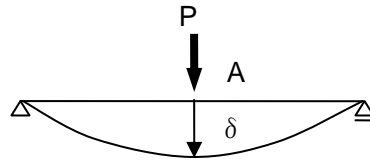
The relation between nodal displacement and element deformation is then expressed as,

$$v(x) = \delta \sin\left(\frac{\pi}{L} x\right) \quad (3)$$

Step.4: Stiffness equation at the node

We obtain the relation between the nodal force and the nodal displacement, for example, by using the “Principle of Virtual Work Method.”

$$P = K\delta \tag{4}$$



The process is summarized as follows:

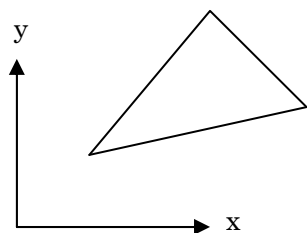
	<p>Translate external forces into equivalent nodal force, P.</p>
	<p>Calculate nodal displacement, δ, from the stiffness equation, $\delta = K^{-1}P$</p>
	<p>Obtain the element deformation from the nodal displacement. $v(x) = \delta \sin\left(\frac{\pi}{L}x\right)$</p>

The above example tells the essence of the finite element analysis, which is: “Assume the deformation pattern to reduce the degree of freedom of the element, then, obtain the deformation from the limited number of nodal displacements.”

3. TRIANGULAR ELEMENT FOR PLANE ANALYSIS

Step.1: Description of the Problem

The problem is to obtain the deformation of a simple triangular element.



There are infinite variations for the deformation patterns.

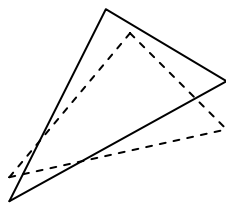


Step.2: Assumption of deformation function

To fix the variation for the deformation patterns, we assume a linear function for the deformation pattern.

$$\begin{aligned} u(x, y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x, y) &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \tag{5}$$

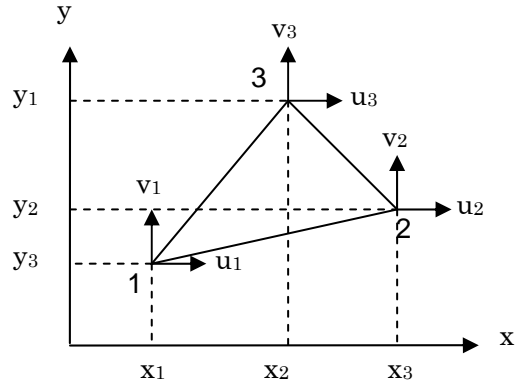
In a matrix form,



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \tag{6}$$

Step.3: Relation between nodal displacement and element deformation

The displacements of the element nodes are expressed as,



$$\text{Node 1: } \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \quad (7)$$

$$\text{Node 2: } \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \quad (8)$$

$$\text{Node 3: } \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \quad (9)$$

It is summarized as,

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \quad (10)$$

$$\mathbf{u} = \mathbf{A} \boldsymbol{\alpha}$$

We can obtain the coefficients $\alpha_1, \dots, \alpha_6$ from the nodal displacements as,

$$\alpha = A^{-1} u \quad (11)$$

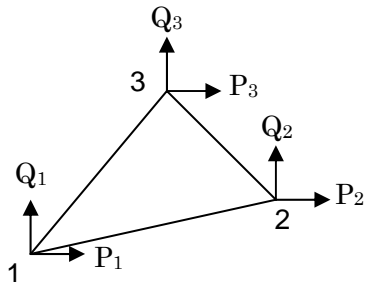
Substituting equation (11) into equation (6), the relation between nodal displacement and element deformation is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} A^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (12)$$

$$u(x,y) = H(x,y) u$$

Step.4: Stiffness equation at the node

We obtain the relation between the nodal force and the nodal displacement, for example, by using the “Principle of Virtual Work Method.”



$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = K \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (13)$$

$$F = K u$$

The process is summarized as follows:

- (1) Translate external forces into equivalent nodal force,
 $F = \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}^T$
- (2) Calculate the nodal displacements from the stiffness equation,
 $u = K^{-1} F$
- (3) Obtain the element deformation from the nodal displacement.
 $u(x,y) = H(x,y)u$

4. STIFFNESS MATRIX FOR TRIANGULAR ELEMENT

Stiffness matrix in equation (13) can be obtained from the “*Principle of Virtual Work Method*,” which is expressed in the following form:

$$\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\sigma} dv = \bar{\boldsymbol{u}}^T \boldsymbol{F} \quad (14)$$

where, $\bar{\boldsymbol{\varepsilon}}$ is a virtual strain vector, $\boldsymbol{\sigma}$ is a stress vector, $\bar{\boldsymbol{u}}$ is a virtual displacement vector and \boldsymbol{F} is a load vector, respectively.

In case of the plane problem, the strain $\boldsymbol{\varepsilon}$ vector is defined as,

$$\begin{pmatrix} \boldsymbol{\varepsilon}_x \\ \boldsymbol{\varepsilon}_y \\ \boldsymbol{\gamma}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \quad (15)$$

Substituting equation (12) into equation (15), the strain vector is calculated from the nodal displacement vector as,

$$\begin{pmatrix} \boldsymbol{\varepsilon}_x \\ \boldsymbol{\varepsilon}_y \\ \boldsymbol{\gamma}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \boldsymbol{A}^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (16)$$

$$\boldsymbol{\varepsilon} = \boldsymbol{B} \boldsymbol{u}$$

In the plane stress problem, the stress-strain relationship is expressed as,

$$\begin{pmatrix} \boldsymbol{\sigma}_x \\ \boldsymbol{\sigma}_y \\ \boldsymbol{\tau}_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_x \\ \boldsymbol{\varepsilon}_y \\ \boldsymbol{\gamma}_{xy} \end{pmatrix} \quad (17)$$

$$\boldsymbol{\sigma} = \boldsymbol{C} \boldsymbol{\varepsilon}$$

Substituting equation (16) into equation (17),

$$\sigma = \mathbf{C} \mathbf{B} \mathbf{u} \quad (18)$$

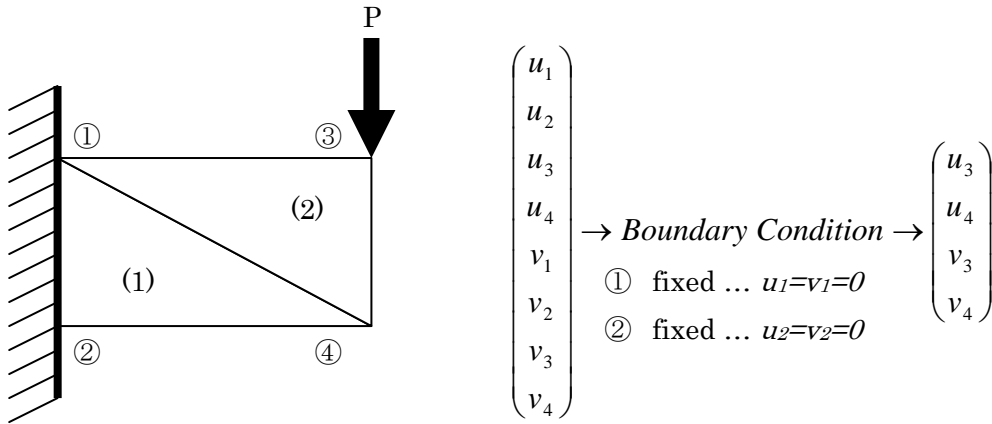
From the Principle of Virtual Work Method,

$$\int_v (\mathbf{B}\bar{\mathbf{u}})^T (\mathbf{C}\mathbf{B}\mathbf{u}) dv = \bar{\mathbf{u}}^T \left(\int_v \mathbf{B}^T \mathbf{C} \mathbf{B} dv \right) \mathbf{u} = \bar{\mathbf{u}}^T \mathbf{F} \quad (19)$$

Therefore, the stiffness equation is obtained as,

$$\mathbf{F} = \mathbf{K}\mathbf{u}, \quad \mathbf{K} = \int_v \mathbf{B}^T \mathbf{C} \mathbf{B} dv \quad (20)$$

5. FROM ELEMENT STIFFNESS MATRIX TO GLOBAL STIFFNESS MATRIX



Element Stiffness Matrix:

$$\text{Element (1) ...} \begin{pmatrix} P_1 \\ P_2 \\ P_4 \\ Q_1 \\ Q_2 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & k_{15}^{(1)} & k_{16}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} & k_{25}^{(1)} & k_{26}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} & k_{35}^{(1)} & k_{36}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} & k_{45}^{(1)} & k_{46}^{(1)} \\ k_{51}^{(1)} & k_{52}^{(1)} & k_{53}^{(1)} & k_{54}^{(1)} & k_{55}^{(1)} & k_{56}^{(1)} \\ k_{61}^{(1)} & k_{62}^{(1)} & k_{63}^{(1)} & k_{64}^{(1)} & k_{65}^{(1)} & k_{66}^{(1)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_4 \\ v_1 \\ v_2 \\ v_4 \end{pmatrix} \quad (21)$$

$$\text{Element (2) ...} \begin{pmatrix} P_1 \\ P_3 \\ P_4 \\ Q_1 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} & k_{15}^{(2)} & k_{16}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} & k_{45}^{(2)} & k_{46}^{(2)} \\ k_{51}^{(2)} & k_{52}^{(2)} & k_{53}^{(2)} & k_{54}^{(2)} & k_{55}^{(2)} & k_{56}^{(2)} \\ k_{61}^{(2)} & k_{62}^{(2)} & k_{63}^{(2)} & k_{64}^{(2)} & k_{65}^{(2)} & k_{66}^{(2)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \\ u_4 \\ v_1 \\ v_3 \\ v_4 \end{pmatrix} \quad (22)$$

Global Stiffness Matrix:

$$\text{Element (2) ...} \begin{pmatrix} P_3 \\ P_4 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{22}^{(2)} & k_{23}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ k_{32}^{(2)} & k_{33}^{(1)} + k_{33}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} + k_{36}^{(2)} \\ k_{52}^{(2)} & k_{53}^{(2)} & k_{55}^{(2)} & k_{56}^{(2)} \\ k_{62}^{(2)} & k_{63}^{(1)} + k_{63}^{(2)} & k_{65}^{(2)} & k_{66}^{(1)} + k_{66}^{(2)} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \\ v_3 \\ v_4 \end{pmatrix} \quad (23)$$

$$\mathbf{F} = \mathbf{K} \mathbf{U}$$

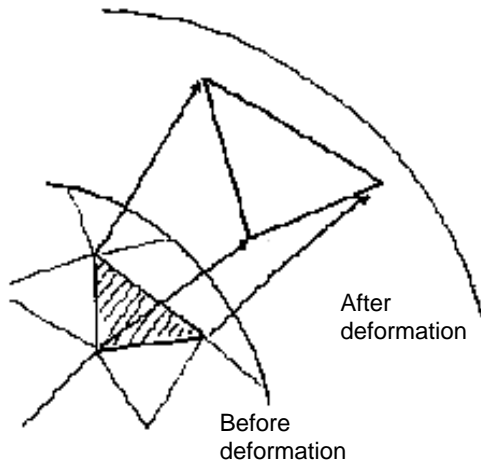
Set the load condition,

$$\begin{pmatrix} P_3 \\ P_4 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -P \\ 0 \end{pmatrix} \quad (24)$$

The displacement vector is then obtained by solving the stiffness equation,

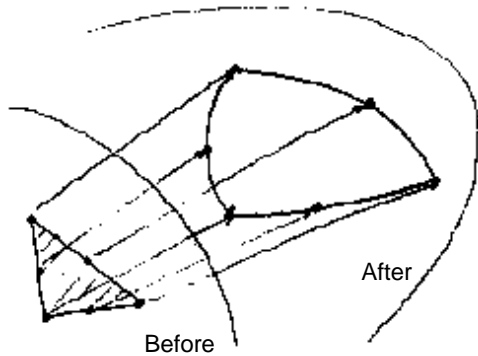
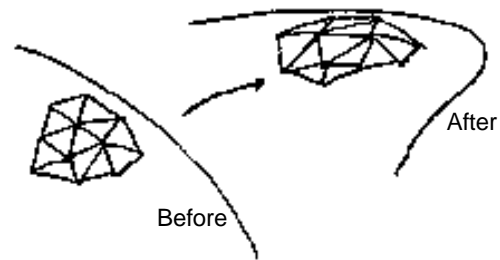
$$\begin{pmatrix} u_3 \\ u_4 \\ v_3 \\ v_4 \end{pmatrix} = K^{-1} \begin{pmatrix} 0 \\ 0 \\ -P \\ 0 \end{pmatrix} \quad (25)$$

6. HIGHER ORDER ELEMENT



The linear triangular element assumes the deformation pattern to be a linear function between two nodes.

It requires a large number of elements at the place where deformation changes largely.

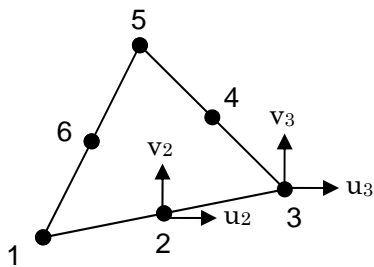


To reduce the number of elements, we introduce the higher order elements, such as the following second order elements where the deformation pattern is assumed to be the second order function of coordinate.

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \\ v &= \alpha_7 + \alpha_8 x + \alpha_9 y + \alpha_{10} x^2 + \alpha_{11} xy + \alpha_{12} y^2 \end{aligned} \quad (26)$$

In a matrix form,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{12} \end{pmatrix} \quad (27)$$



In order to define the second order function, we need an additional node in the middle of each side of the triangle. At the result, the total number of nodes in one element is 6.

The displacement of the element nodes are then expressed as,

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ - \\ v_1 \\ v_2 \\ \vdots \\ v_6 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & | & & & & & & \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & | & & & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & | & & & & & & \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 & | & & & & & & \\ - & - & - & - & - & - & | & - & - & - & - & - & - \\ & & & & & & | & 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ & & & & & & | & 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ & & & & & & | & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & | & 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_6 \\ - \\ \alpha_7 \\ \alpha_8 \\ \vdots \\ \alpha_{12} \end{pmatrix} \quad (28)$$

$$\mathbf{u} = \mathbf{A} \boldsymbol{\alpha}$$

From equations (27) and (28), we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{pmatrix} A^{-1} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ v_2 \\ \vdots \\ v_6 \end{pmatrix} \quad (29)$$

$$\mathbf{u}(x,y) = \mathbf{H}(x,y) \mathbf{u}$$

As the same as the linear triangular element, the stiffness equation is obtained as

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_6 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_6 \end{pmatrix} = \mathbf{K} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ v_2 \\ \vdots \\ v_6 \end{pmatrix} \quad (30)$$

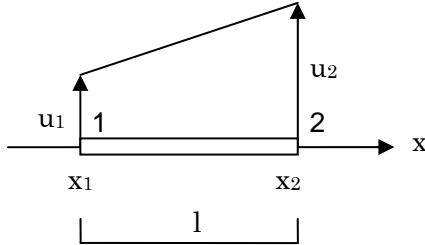
$$\mathbf{F} = \mathbf{K} \mathbf{u}$$

The process is summarized as follows:

- (1) Translate external forces into equivalent nodal force,
 $\mathbf{F} = \{P_1, \dots, P_6, Q_1, \dots, Q_6\}^T$
- (2) Calculate the nodal displacements from the stiffness equation,
 $\mathbf{u} = \mathbf{K}^{-1} \mathbf{F}$
- (3) Obtain the element deformation from the nodal displacement.
 $\mathbf{u}(\mathbf{x}, \mathbf{y}) = \mathbf{H}(\mathbf{x}, \mathbf{y}) \mathbf{u}$

7. INTERPOLATION FUNCTION

Suppose we have one dimensional element under loading. As discussed before, we assume a linear function for the deformation pattern after loading,



or

$$u(x) = a_0 + a_1 x$$

$$u(x) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad (31)$$

The next step is to obtain the coefficients, a_0 , a_1 , from the nodal displacements. From the relations:

$$u_1 = a_0 + a_1 x_1$$

$$u_2 = a_0 + a_1 x_2$$

or

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad (32)$$

$$\mathbf{u} = \mathbf{A} \boldsymbol{\alpha}$$

The coefficients are obtained as, $\boldsymbol{\alpha} = \mathbf{A}^{-1} \mathbf{u}$. Then, the relation between the deformation and the nodal displacements is,

$$u(x) = \begin{pmatrix} 1 & x \end{pmatrix} \mathbf{A}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (33)$$

Instead of the previous procedure, we introduce the **interpolation functions** to express the deformation directly from the nodal displacements:

$$u(x) = h_1(x)u_1 + h_2(x)u_2 \quad (34)$$

The interpolation functions, h_1 and h_2 , have the following characteristics:

$$h_1(x) = \begin{cases} 1, & x = u_1 \\ 0, & x \neq u_1 \end{cases}, \quad h_2(x) = \begin{cases} 1, & x = u_2 \\ 0, & x \neq u_2 \end{cases} \quad (35)$$

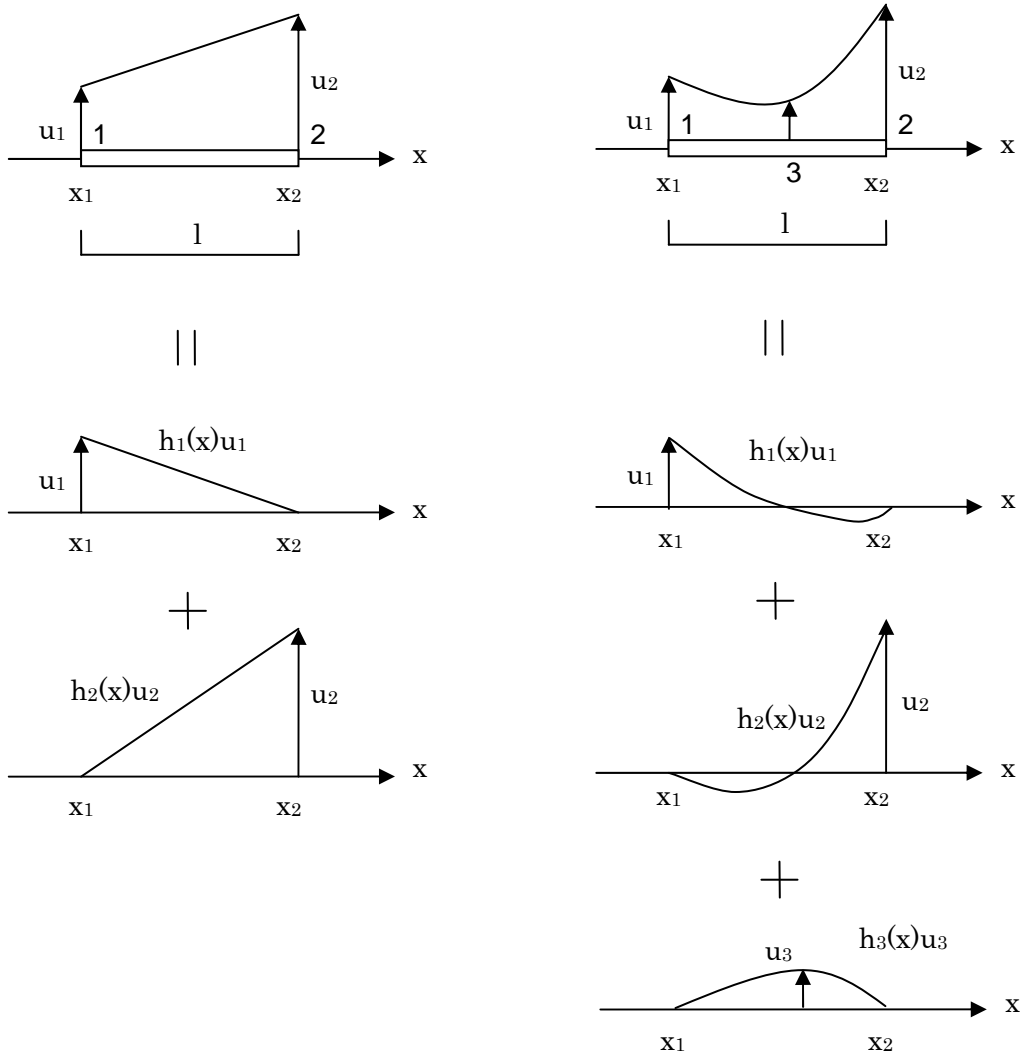
From these characteristics, the functions are easily obtained as,

$$h_1(x) = \frac{x_2 - x}{l}, \quad h_2(x) = \frac{x - x_1}{l} \quad (36)$$

One of the advantages of using interpolation functions is to reduce the burden to calculate the inverse matrix of A in equation (33).

In the same manner, if we assume a second order function for the deformation pattern, the deformation can be directly expressed using interpolation functions as follows:

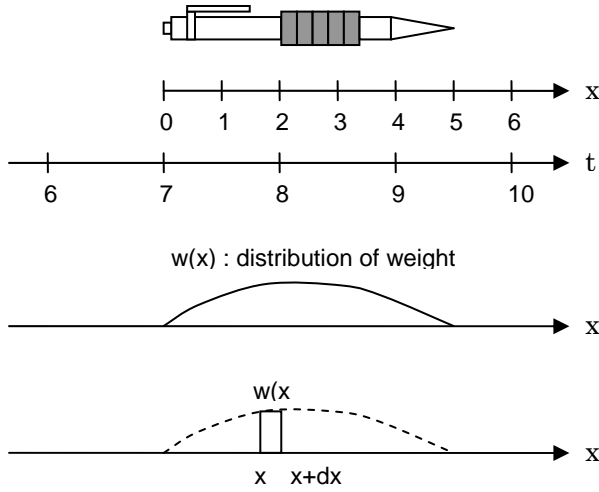
$$u(x) = h_1(x)u_1 + h_2(x)u_2 + h_3(x)u_3 \quad (37)$$



First order interpolation function

Second order interpolation function

8. NATURAL COORDINATE



When we measure the coordinate of the pencil, the result is different depending on the scale we use. In this example, the coordinate of the head of the pencil is 5.0 in x-scale and 9.5 in t-scale.

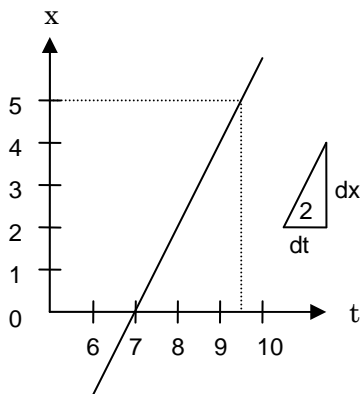
As long as we have one-to-one relationship between two scales, we can translate the value in one scale to the value in another scale anytime.

$$\begin{array}{ccc} & \xrightarrow{t = 7 + 0.5x} & \\ x & & t \\ & \xleftarrow{x = 2(t - 7)} & \end{array}$$

The total weight of the pencil will be calculated in x-axis as,

$$W = \int_0^5 w(x) dx \quad (38)$$

To translate it into t-axis, we use the following relationships:



Global relationship:

$$x = 2(t - 7) \quad (39)$$

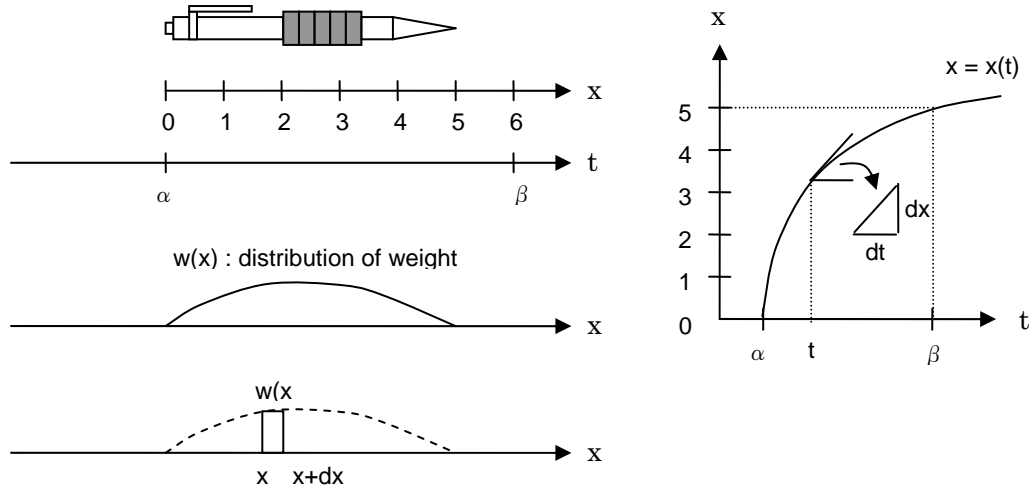
Local relationship:

$$dx = 2dt \quad (40)$$

Substituting equations (39) and (40) into (38), the total weight is expressed in t-axis as,

$$W = 2 \int_{7.5}^{9.5} w(x(t)) dt \quad (41)$$

Next we consider a more complicated scale to measure the total weight of the pencil.



The relationships between x-axis and t-axis are:

$$\text{Global relationship: } x = x(t) \quad (42)$$

$$\text{Local relationship: } dx = \frac{dx(t)}{dt} dt \quad (43)$$

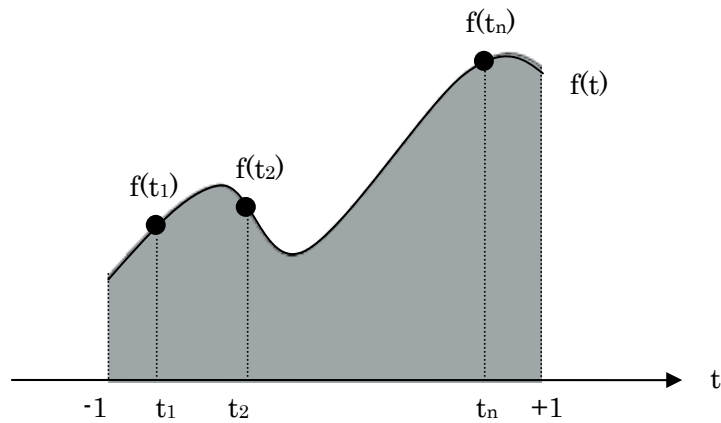
Where $dx(t)/dt$ represents the first derivative of $x(x)$ by the variable t , which correspond to the slope of $x(t)$ at t . Substituting equations (42) and (43) into (38), the total weight will be expressed in t-axis as,

$$W = \int_{\alpha}^{\beta} w(x(t)) \frac{dx(t)}{dt} dt \quad (44)$$

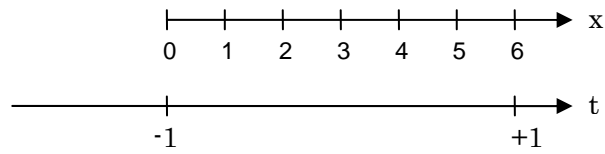
This integration can be evaluated by the numerical integration formula which is generally expressed in the following form:

$$\int_{-1}^1 f(t) dt \approx \alpha_1 f(t_1) + \alpha_2 f(t_2) + \dots + \alpha_n f(t_n) \quad (45)$$

where, $\alpha_1, \dots, \alpha_n$ are the weighting coefficients. This formula requires a limited number of function values, $f(t_1), \dots, f(t_n)$, at the sampling points, t_1, \dots, t_n , to evaluate the integration. The integration range must be $[-1, 1]$ to use this formula.



Going back to the problem to evaluate the total weight of the pencil, if we select the scale which has the range $[-1, 1]$, we can use the numerical integration method to evaluate the integration.



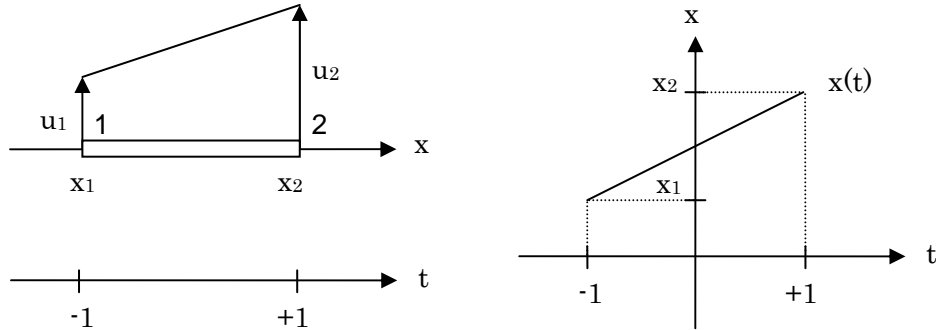
Setting $\alpha = -1$, $\beta = 1$ in equation (44),

$$W = \int_{-1}^{+1} f(t) dt, \quad f(t) = w(x(t)) \frac{dx(t)}{dt} \quad (46)$$

Such coordinate is called “**natural coordinate.**”

9. ISOPARAMETRIC ELEMENT

We now introduce the natural coordinate for the example of one dimensional element.



If we assume the linear transfer function $x(t)$ between x -axis and t -axis, $x(t)$ will be expressed as

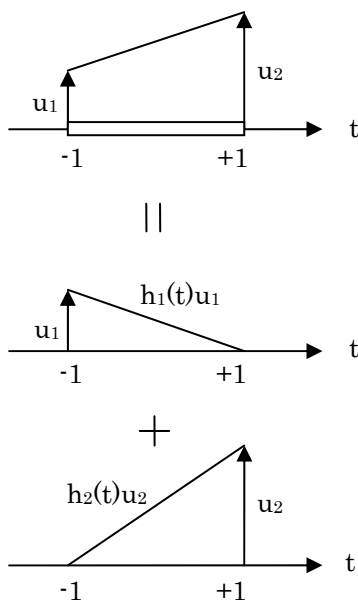
$$x(t) = h_1(t)x_1 + h_2(t)x_2 \quad (47)$$

where

$$h_1(t) = \frac{1}{2}(1-t), \quad h_2(t) = \frac{1}{2}(1+t) \quad (48)$$

Actually, it satisfies the fact that

$$x(-1) = x_1, \quad x(1) = x_2 \quad (49)$$



The deformation of the element is also expressed as,

$$u(t) = h_1(t)u_1 + h_2(t)u_2 \quad (50)$$

Therefore, the functions $h_1(t)$, $h_2(t)$ are the interpolation functions we introduced before.

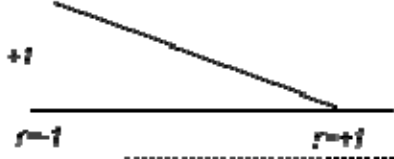
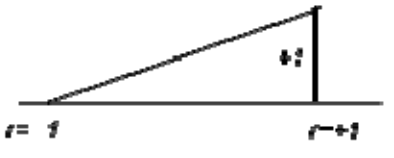
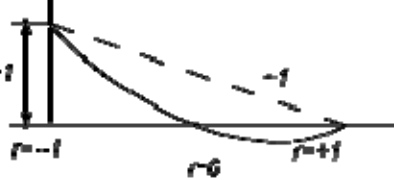

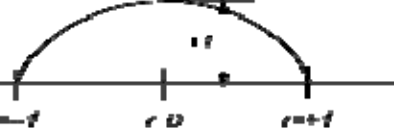
The element where both the coordinate transfer function $x(t)$ and the deformation function $u(t)$ are expressed using the same interpolation functions on the natural coordinate is called “**Isoparametric element.**”

Advantages of using isplarametric elements are summarized below:

- (1) The relation $u(t) = \sum_{i=1}^n h_i(t)u_i$ does not require the calculation of inverse matrix.
- (2) The relation $x(t) = \sum_{i=1}^n h_i(t)x_i$ enables to use the numerical integration method.
- (3) Both functions $u(t)$ and $x(t)$ are expressed using the same interpolation functions.

10. SYSTEMATIC FORMULATION OF INTERPOLATION FUNCTION

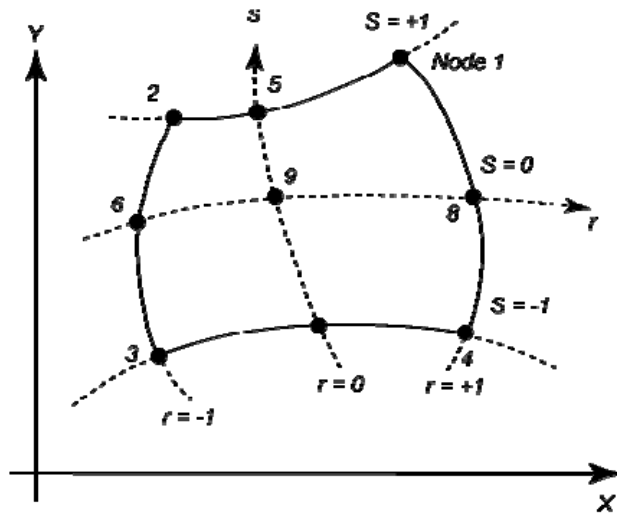
(1) One dimensional element

2Nodes		$h_1 = \frac{1}{2}(1-r)$
		$h_2 = \frac{1}{2}(1+r)$
3Nodes		$h_1 = \frac{1}{2}(1-r) - \frac{1}{2}(1-r^2)$
		$h_2 = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2)$
		$h_3 = 1-r^2$

As presented here, if you increase a node to define the second order function for the deformation, the interpolation function changes in the following manners:

- Modify the existing interpolation functions, h_1 and h_2 ,
- Define a new interpolation function, h_3 .

(2) Two dimensional element



(a) Four to 9 variable-number-nodes two-dimensional element

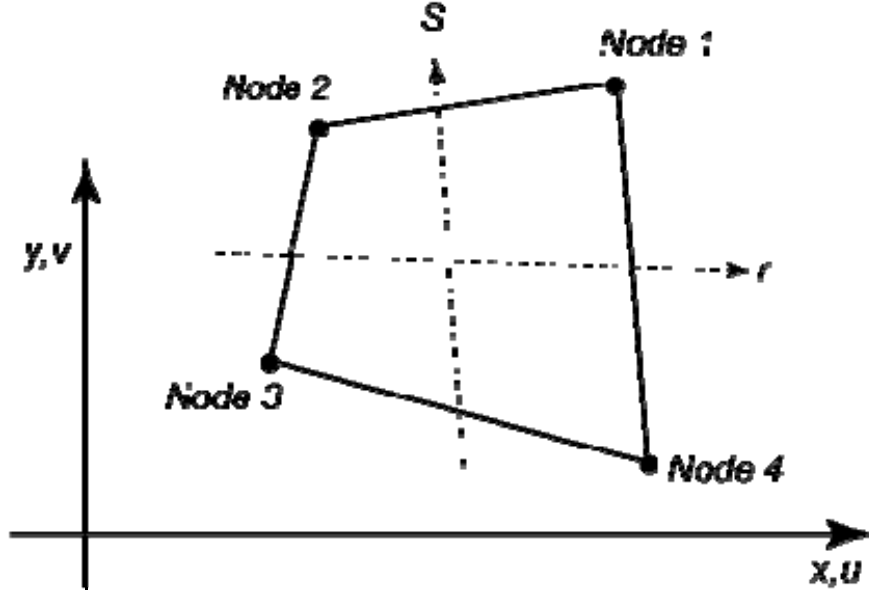
Include inly if node i is defined

$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$h_1 = \frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{2}h_9$
$h_2 = \frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{2}h_9$
$h_3 = \frac{1}{4}(1-r)(1+s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{2}h_9$
$h_4 = \frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{2}h_9$
$h_5 = \frac{1}{4}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 = \frac{1}{4}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 = \frac{1}{4}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 = \frac{1}{4}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 = (1-r^2)(1-s^2)$					

(b) Interpolation functions

11. STIFFNESS MATRIX FOR ISOPARAMETRIC ELEMENT

Using a two dimensional isoparametric element, we will see the procedure to derive the stiffness matrix.



The coordinate transfer function $\{x, y\}$ is expressed using the interpolation functions as follows:

$$\begin{aligned}
 x(r, s) &= \sum_{i=1}^4 h_i(r, s)x_i = \frac{1}{4}(1+r)(1+s)x_1 + \frac{1}{4}(1-r)(1+s)x_2 + \frac{1}{4}(1-r)(1-s)x_3 + \frac{1}{4}(1+r)(1-s)x_4 \\
 y(r, s) &= \sum_{i=1}^4 h_i(r, s)y_i = \frac{1}{4}(1+r)(1+s)y_1 + \frac{1}{4}(1-r)(1+s)y_2 + \frac{1}{4}(1-r)(1-s)y_3 + \frac{1}{4}(1+r)(1-s)y_4
 \end{aligned}
 \tag{51}$$

The deformation function $\{u, v\}$ is also expressed using the same interpolation functions.

$$\begin{aligned}
 u(r, s) &= \sum_{i=1}^4 h_i(r, s)u_i = \frac{1}{4}(1+r)(1+s)u_1 + \frac{1}{4}(1-r)(1+s)u_2 + \frac{1}{4}(1-r)(1-s)u_3 + \frac{1}{4}(1+r)(1-s)u_4 \\
 v(r, s) &= \sum_{i=1}^4 h_i(r, s)v_i = \frac{1}{4}(1+r)(1+s)v_1 + \frac{1}{4}(1-r)(1+s)v_2 + \frac{1}{4}(1-r)(1-s)v_3 + \frac{1}{4}(1+r)(1-s)v_4
 \end{aligned}
 \tag{52}$$

Stiffness matrix can be obtained from the “*Principle of Virtual Work Method*,” which is expressed in the following form:

$$\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\sigma} \, dv = \bar{\boldsymbol{u}}^T \boldsymbol{F} \quad (53)$$

where, $\bar{\boldsymbol{\varepsilon}}$ is a virtual strain vector, $\boldsymbol{\sigma}$ is a stress vector, $\bar{\boldsymbol{u}}$ is a virtual displacement vector and \boldsymbol{F} is a load vector, respectively.

In case of the plane problem, the strain $\boldsymbol{\varepsilon}$ vector is defined as,

$$\begin{pmatrix} \boldsymbol{\varepsilon}_x \\ \boldsymbol{\varepsilon}_y \\ \boldsymbol{\gamma}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \quad (54)$$

Substituting equation (52) into equation (54), the strain vector is calculated from the nodal displacement vector as,

$$\begin{aligned} \begin{pmatrix} \boldsymbol{\varepsilon}_x \\ \boldsymbol{\varepsilon}_y \\ \boldsymbol{\gamma}_{xy} \end{pmatrix} &= \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^4 \frac{\partial h_i}{\partial x} u_i \\ \sum_{i=1}^4 \frac{\partial h_i}{\partial y} v_i \\ \sum_{i=1}^4 \frac{\partial h_i}{\partial y} u_i + \sum_{i=1}^4 \frac{\partial h_i}{\partial x} v_i \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 & \frac{\partial h_3}{\partial x} & 0 & \frac{\partial h_4}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} & 0 & \frac{\partial h_3}{\partial y} & 0 & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} \\ \boldsymbol{\varepsilon} &= \boldsymbol{B} \boldsymbol{u} \quad (55) \end{aligned}$$

In the plane stress problem, the stress-strain relationship is expressed as,

$$\begin{pmatrix} \boldsymbol{\sigma}_x \\ \boldsymbol{\sigma}_y \\ \boldsymbol{\tau}_{xy} \end{pmatrix} = \frac{E}{1-\nu} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_x \\ \boldsymbol{\varepsilon}_y \\ \boldsymbol{\gamma}_{xy} \end{pmatrix} \quad (56)$$

$$\boldsymbol{\sigma} = \boldsymbol{C} \boldsymbol{\varepsilon}$$

Substituting equation (55) into equation (56),

$$\sigma = C B u \quad (57)$$

From the Principle of Virtual Work Method,

$$\int_V (B\bar{u})^T (CBu) dv = \bar{u}^T \left(\int_{V(x,y)} B^T CB dx dy \right) u = \bar{u}^T F \quad (58)$$

Therefore, the stiffness equation is obtained as,

$$F = Ku, \quad K = \int_V B^T CB dv \quad (59)$$

If we assume the constant thickness of the plate (= t), using the relation $dv = t dx dy$,

$$K = t \int_{V(x,y)} B^T CB dx dy \quad (60)$$

Since this integration is defined in x-y coordinate, we must transfer the coordinate into r-s coordinate to use the numerical integration method. Introducing the **Jacobian matrix**,

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix}; \text{ Jacobian Matrix} \quad (61)$$

the above integration is expressed in r-s coordinate as,

$$K = t \int_{-1}^1 \int_{-1}^1 B(x(r,s), y(r,s))^T CB(x(r,s), y(r,s)) \frac{\partial(x,y)}{\partial(r,s)} dr ds \quad (62)$$

where

$$\frac{\partial(x, y)}{\partial(r, s)} = \det J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} \quad (63)$$

1) Evaluation of Jacobian Matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^4 \frac{\partial h_i}{\partial r} x_i & \sum_{i=1}^4 \frac{\partial h_i}{\partial r} y_i \\ \sum_{i=1}^4 \frac{\partial h_i}{\partial s} x_i & \sum_{i=1}^4 \frac{\partial h_i}{\partial s} y_i \end{pmatrix} \quad (64)$$

2) Evaluation of the matrix B

$$B = \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 & \frac{\partial h_3}{\partial x} & 0 & \frac{\partial h_4}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} & 0 & \frac{\partial h_3}{\partial y} & 0 & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x} \end{pmatrix} \quad (65)$$

The derivatives $\frac{\partial h_1}{\partial x}, \dots, \frac{\partial h_4}{\partial x}, \frac{\partial h_1}{\partial y}, \dots, \frac{\partial h_4}{\partial y}$ are calculated as,

$$\begin{aligned} \frac{\partial h_1}{\partial x} &= \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial h_1}{\partial s} \frac{\partial s}{\partial x}, \quad \dots, \quad \frac{\partial h_4}{\partial x} = \frac{\partial h_4}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial h_4}{\partial s} \frac{\partial s}{\partial x}, \\ \frac{\partial h_1}{\partial y} &= \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h_1}{\partial s} \frac{\partial s}{\partial y}, \quad \dots, \quad \frac{\partial h_4}{\partial y} = \frac{\partial h_4}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h_4}{\partial s} \frac{\partial s}{\partial y} \end{aligned}$$

In a matrix form,

$$\begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial x} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_2}{\partial y} & \frac{\partial h_3}{\partial y} & \frac{\partial h_4}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_2}{\partial r} & \frac{\partial h_3}{\partial r} & \frac{\partial h_4}{\partial r} \\ \frac{\partial h_1}{\partial s} & \frac{\partial h_2}{\partial s} & \frac{\partial h_3}{\partial s} & \frac{\partial h_4}{\partial s} \end{pmatrix}$$

$$= J^{-1} \begin{pmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_2}{\partial r} & \frac{\partial h_3}{\partial r} & \frac{\partial h_4}{\partial r} \\ \frac{\partial h_1}{\partial s} & \frac{\partial h_2}{\partial s} & \frac{\partial h_3}{\partial s} & \frac{\partial h_4}{\partial s} \end{pmatrix} \quad (66)$$

3) Evaluation of partial derivatives of the interpolation functions

$$\begin{aligned} \frac{\partial h_1}{\partial r} &= \frac{1}{4}(1+s) & \frac{\partial h_1}{\partial s} &= \frac{1}{4}(1+r) \\ \frac{\partial h_2}{\partial r} &= -\frac{1}{4}(1+s) & \frac{\partial h_2}{\partial s} &= \frac{1}{4}(1-r) \\ \frac{\partial h_3}{\partial r} &= -\frac{1}{4}(1-s) & \frac{\partial h_3}{\partial s} &= -\frac{1}{4}(1-r) \\ \frac{\partial h_4}{\partial r} &= \frac{1}{4}(1-s) & \frac{\partial h_4}{\partial s} &= -\frac{1}{4}(1+s) \end{aligned} \quad (67)$$

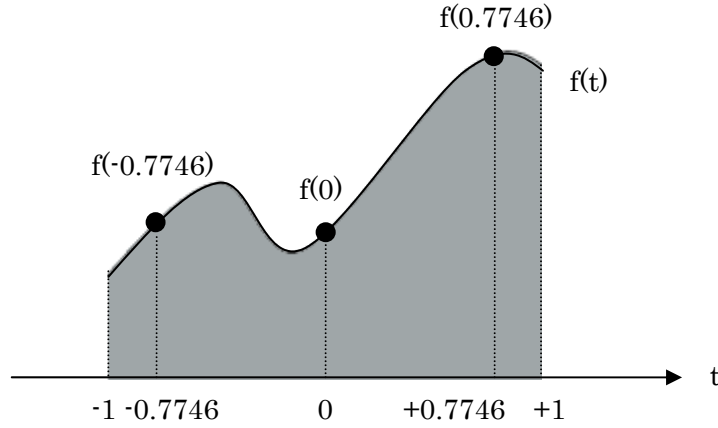
12. NUMERICAL INTEGRATION METHOD

The **3 points Gauss Integration Formula** is defined as:

$$\int_{-1}^1 f(t)dt = 0.5556 f(-0.7746) + 0.8889 f(0) + 0.5556 f(0.7746) \quad (68)$$

$$= \alpha_1 f(t_1) + \alpha_2 f(t_2) + \alpha_3 f(t_3)$$

where, $\alpha_1 = 0.5556, \alpha_2 = 0.8889, \alpha_3 = 0.5556$
 $t_1 = -0.7746, t_2 = 0, t_3 = 0.7746$



The stiffness matrix is then calculated numerically as follows:

$$K = t \int_{-1}^1 \int_{-1}^1 B(x(r,s), y(r,s))^T CB(x(r,s), y(r,s)) \frac{\partial(x,y)}{\partial(r,s)} drds$$

$$= t \int_{-1}^1 \int_{-1}^1 F(r,s) drds \quad (69)$$

$$= t \sum_{i=1}^3 \sum_{j=1}^3 \alpha_i \alpha_j F(r_i, s_j)$$

where

$$F(r,s) = B(x(r,s), y(r,s))^T CB(x(r,s), y(r,s)) \frac{\partial(x,y)}{\partial(r,s)}$$

$$\alpha_1 = 0.5556, \alpha_2 = 0.8889, \alpha_3 = 0.5556$$

$$r_1 = s_1 = -0.7746, r_2 = s_2 = 0, r_3 = s_3 = 0.7746$$