2.4 Travel-time inversion

- Time-term method
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  - Velocity gradient
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- 2D and 3D Travel-time inversion
  - Least squares inversion
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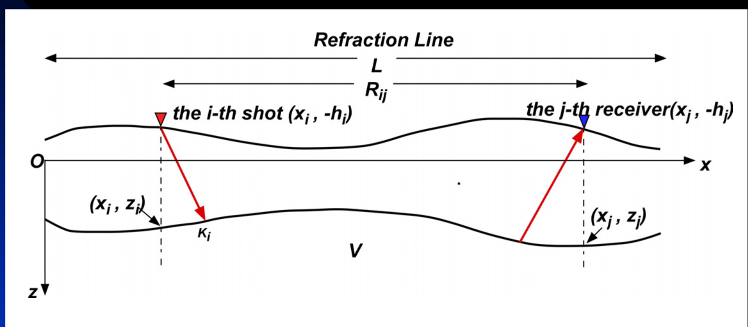


Basic equation

$$t_{ij} = a_i + b_j + R_{ij} / V.$$

a_i, b_j : time-terms defined by

$$a_i = \sqrt{v_{1i}^{-2} - V^{-2}} H_{1i} \qquad (i = 1, .., m_s),$$

$$b_j = \sqrt{v_{1j}^{-2} - V^{-2}} H_{1j} \qquad (j = 1, .., m_r).$$

Inversion scheme

$$\mathbf{t} = \mathbf{Am},$$

where

$$\mathbf{t}^{\mathrm{T}} = (t_{11}, t_{12}, ..., t_{m_s m_r}),$$

$$\mathbf{m}^{\mathrm{T}} = (a_1, a_2, .., a_{m_s}; b_1, b_2, .., b_{m_r}; V^{-1}).$$

with

$$A_{pq} = \begin{cases} 1 & (q = i_p \ \ or \ \ q = m_s + j_p), \\ R_{i_p j_p} & (q = m_s + m_r + 1), \\ 0 & \text{otherwise.} \end{cases}.$$

Prior constraints between time-terms

$$a_{i_p} - b_{j_p} = \beta_p \quad (p = 1, \ldots, n_c).$$

$$\rightarrow \quad \mathbf{c} = \mathbf{B}\mathbf{m}$$

with $\mathbf{c}^{\mathrm{T}} = (\beta_1, \beta_2, \ldots, \beta_{n_c}),$

$$B_{pq} = \left\{ \begin{array}{ll} 1 & (1 \le p \le n_c,\ q = i_p), \\ -1 & (1 \le p \le n_c,\ q = m_s + j_p), \\ 0 & \text{(otherwise)} \end{array} \right\}$$

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \mathbf{m}.$$

Taking

$$\mathbf{d} = \begin{bmatrix} \mathbf{t} \\ \mathbf{c} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix},$$

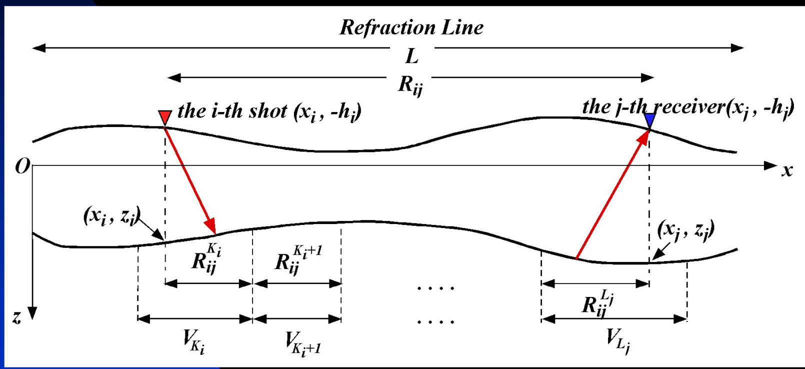
$$\mathbf{d} = \mathbf{C}\mathbf{m}$$

Least squares solution

$$\mathbf{C}^{\mathrm{T}}\mathbf{d} = \mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{m} \ \rightarrow \ \mathbf{m} = (\mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{d}.$$

Damped least squares solution

$$\mathbf{m} = (\mathbf{C}^{\mathrm{T}}\mathbf{C} + \alpha\mathbf{I})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{d}.$$



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Basic equation

$$t_{ij} = a_i + b_j + \sum_{k=K_i}^{L_j} R_{ij}^k / V_k .$$

$$x = x_1,\ z = x_3.$$

a_i, b_j : time-terms defined by

$$a_i = \int_{-h_i}^{z_i} \sqrt{v^{-2}(x_i, z) - U_i^2}\, dz \qquad (i = 1,..,m_s),$$

$$b_j = \int_{-h'_j}^{z'_j} \sqrt{v^{-2}(x'_j, z) - U'^2_j}\, dz \quad (j = 1,..,m_r).$$

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Inversion scheme

$$\mathbf{t} = \mathbf{Am} + \mathbf{e},$$

where

$$\mathbf{t}^{\mathrm{T}} = (t_{11}, t_{12}, \ldots, t_{m_s m_r}),$$

$$\mathbf{m}^{\mathrm{T}} = (a_1, a_2, \ldots, a_{m_s}; b_1, b_2, \ldots, b_{m_r}; V_1^{-1}, V_2^{-1}, \ldots, V_{m_v}^{-1}).$$

and

$$\mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{E}).$$

Stochastic model for **m** and **t**

$$p(\mathbf{t} \mid \mathbf{m}; \sigma^2) = (2\pi\sigma^2)^{-n/2} |\mathbf{E}|^{-1/2} \times \exp\left[-\frac{1}{2\sigma^2} (\mathbf{t} - \mathbf{Am})^T \mathbf{E}^{-1} (\mathbf{t} - \mathbf{Am}) \right].$$

Prior constraints : $a_{i_p} - b_{j_p} = \beta_p \;\; (p = 1, \ldots, n_c).$

Initial guess : $\mathbf{m} = \mathbf{m}_o$.

$$\rightarrow \quad \mathbf{c}_1 = \mathbf{B}_1 \mathbf{m} + \mathbf{d}_1,$$

with $\mathrm{c}_1^{\mathrm{T}} = (\beta_1, \beta_2, \ldots, \beta_{n_c}; m_{o1}, m_{o2}, \ldots, m_{om}),$

and $\mathbf{d}_1 \sim N(\mathbf{0}, \rho_1^2 \mathbf{D}_1).$

$$p(m; \rho_1^2) = (2\pi\rho_1^2)^{-r_1/2} |\mathbf{D}_1|^{-1/2} \exp\left[-\frac{1}{2\rho_1^2} (\mathbf{c}_1 - \mathbf{B}_1\mathbf{m})^T \mathbf{D}_1^{-1} (\mathbf{c}_1 - \mathbf{B}_1\mathbf{m}) \right].$$

Smoothness constraint

$$b_{j-1} - 2b_j + b_{j+1} = 0 \quad (j = 2, \ldots, m_r - 1).$$

$$\rightarrow \; \mathbf{0} = \mathbf{B}_2\mathbf{m} + \mathbf{d}_2,$$

$$\text{with } \mathrm{d}_2 \sim N(0, \rho_2^2 D_2).$$

$$p(\mathbf{m}; \rho_2^2) = (2\pi\rho_2^2)^{-r_2/2} |\mathbf{D}_2|^{-1/2} \exp\left[-\frac{1}{2\rho_2^2} (\mathbf{B}_2\mathbf{m})^T \mathbf{D}_2^{-1} (\mathbf{B}_2\mathbf{m}) \right].$$

Likelihood function

$$l(\mathbf{m}; \sigma^2, \rho_1^2, \rho_2^2 \mid \mathbf{d}) = p(t \mid \mathbf{m}; \sigma^2) p(\mathbf{m}; \rho_1^2) p(\mathbf{m}; \rho_2^2).$$

$$l(\mathbf{m}; \sigma^2, \alpha_1^2, \alpha_2^2 \mid \mathbf{d}) = (2\pi\sigma^2)^{-(n+r_1+r_2)/2} (\alpha_1^2)^{r_1/2} (\alpha_2^2)^{r_2/2} \times |\mathbf{E}|^{-1/2} |\mathbf{D}_1|^{-1/2} |\mathbf{D}_2|^{-1/2} \exp\left[-\frac{1}{2\sigma^2} s(\mathbf{m}) \right],$$

with

$$s(\mathbf{m}) = (\mathbf{t} - \mathbf{A}\mathbf{m})^T \mathbf{E}^{-1} (\mathbf{t} - \mathbf{A}\mathbf{m}) + \alpha_1^2 (\mathbf{c}_1 - \mathbf{B}_1\mathbf{m})^T \mathbf{D}_1^{-1} (\mathbf{c}_1 - \mathbf{B}_1\mathbf{m}) + \alpha_2^2 (\mathbf{B}_2\mathbf{m})^T \mathbf{D}_2^{-1} (\mathbf{B}_2\mathbf{m}).$$

$$\mathbf{m}^* = (\mathbf{A}^T\mathbf{E}^{-1}\mathbf{A} + \alpha_1^2 \mathbf{B}_1^T \mathbf{D}_1^{-1} \mathbf{B}_1 + \alpha_2^2 \mathbf{B}_2^T \mathbf{D}_2^{-1} \mathbf{B}_2)^{-1} \times (\mathbf{A}^T\mathbf{E}^{-1}\mathbf{t} + \alpha_1^2 \mathbf{B}_1^T \mathbf{D}_1^{-1} \mathbf{c}_1).$$

$$A_{pq} = \left\{ \begin{array}{ll} 1 & (q = i_p \ or \ q = m_s + j_p), \\ R_{i_p j_p}^{q-(m_s+m_r)} & (m_s + m_r + K_{i_p} \le q \le m_s + m_r + L_{j_p}), \\ 0 & \text{otherwise.} \end{array} \right\}$$

$$B_{1pq} = \left\{ \begin{array}{ll} 1 & (1 \le p \le n_c, \ q = i_p), \\ -1 & (1 \le p \le n_c, \ q = m_s + j_p), \\ 1 & (n_c + 1 \le p \le n_c + m), \\ 0 & \text{(otherwise)} \end{array} \right\}.$$

$$B_{2pq} = \left\{ \begin{array}{ll} 1 & (q = m_s + p), \\ -2 & (q = m_s + p + 1), \\ 1 & (q = m_s + p + 2), \\ 0 & \text{(otherwise).} \end{array} \right\}.$$

$ABIC = (-2)L(\sigma^2, \alpha_1^2, \alpha_2^2),$

with $L(\sigma^2, \alpha_1^2, \alpha_2^2) = \int l(\mathbf{m}; \sigma^2, \alpha_1^2, \alpha_2^2 \mid \mathbf{d}) d\mathbf{m}.$

Minimizing $ABIC$ requires

$$\partial L(\sigma^2, \alpha_1^2, \alpha_2^2) / \partial \sigma^2 = \partial L(\sigma^2, \alpha_1^2, \alpha_2^2) / \partial \alpha_1^2 = \partial L(\sigma^2, \alpha_1^2, \alpha_2^2) / \partial \alpha_2^2 = 0.$$

$\rightarrow \ \sigma^2 = s(\mathbf{m}^*) / (n + r_1 + r_2 - m),$

and

$$\mathrm{ABIC}(\alpha_1^2, \alpha_2^2) = (n + r_1 + r_2 - m) \log s(\mathbf{m}^*) - r_1 \log(\alpha_1^2) - r_2 \log(\alpha_2^2) + \log \left| \mathbf{A}^{\mathrm{T}} \mathbf{E}^{-1} \mathbf{A} + \alpha_1^2 \mathbf{B}_1^{\mathrm{T}} \mathbf{D}_1^{-1} \mathbf{B}_1 + \alpha_2^2 \mathbf{B}_2^{\mathrm{T}} \mathbf{D}_2^{-1} \mathbf{B}_2 \right| + C.$$

Final solution

$$\hat{\mathbf{m}} = (\mathbf{A}^{T}\mathbf{E}^{-1}\mathbf{A} + \hat{\alpha}_1^2 \mathbf{B}_1^{T}\mathbf{D}_1^{-1}\mathbf{B}_1 + \hat{\alpha}_2^2 \mathbf{B}_2^{T}\mathbf{D}_2^{-1}\mathbf{B}_2)^{-1} \times (\mathbf{A}^{T}\mathbf{E}^{-1}\mathbf{t} + \hat{\alpha}_1^2 \mathbf{B}_1^{T}\mathbf{D}_1^{-1}\mathbf{c}_1),$$

and

$$\hat{\sigma}^2 = s(\hat{\mathbf{m}})/(n + r_1 + r_2 - m).$$