

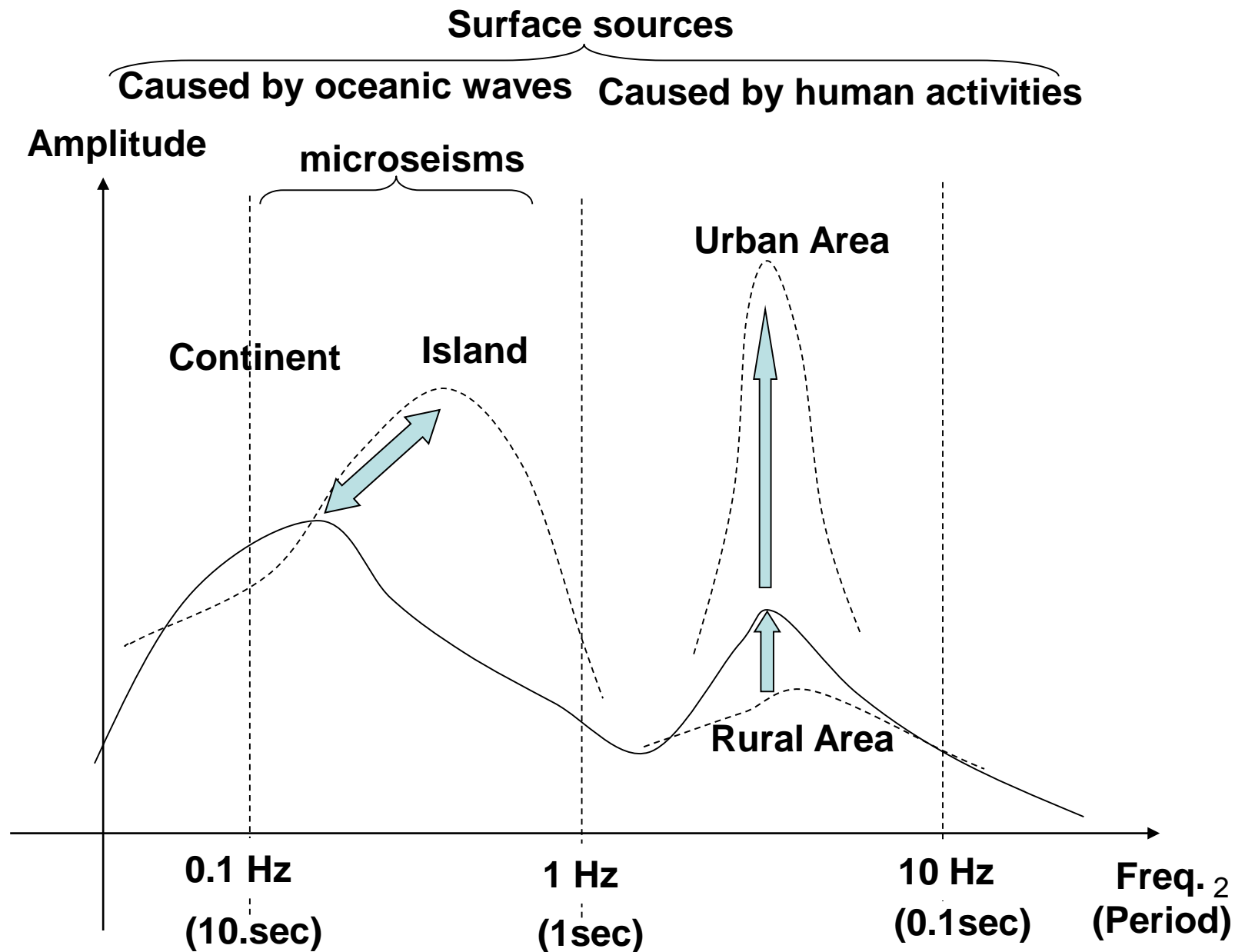
Basic Theory

for CCA Method

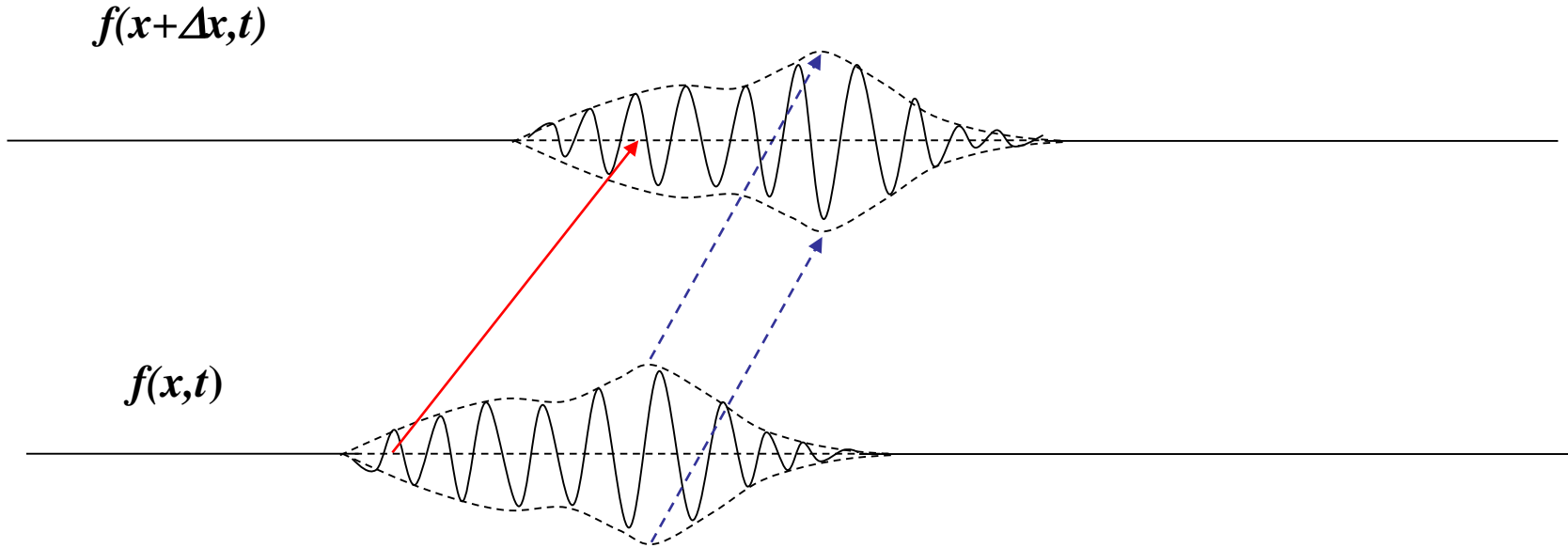
May 11, 2012
IISEE, BRI, Japan

By T.Yokoi

Microtremor (Ambient Noise)



Preparation: Phase Velocity & Group Velocity



Propagation of Energy: Group Velocity v .

Propagation of Information: Phase Velocity c .

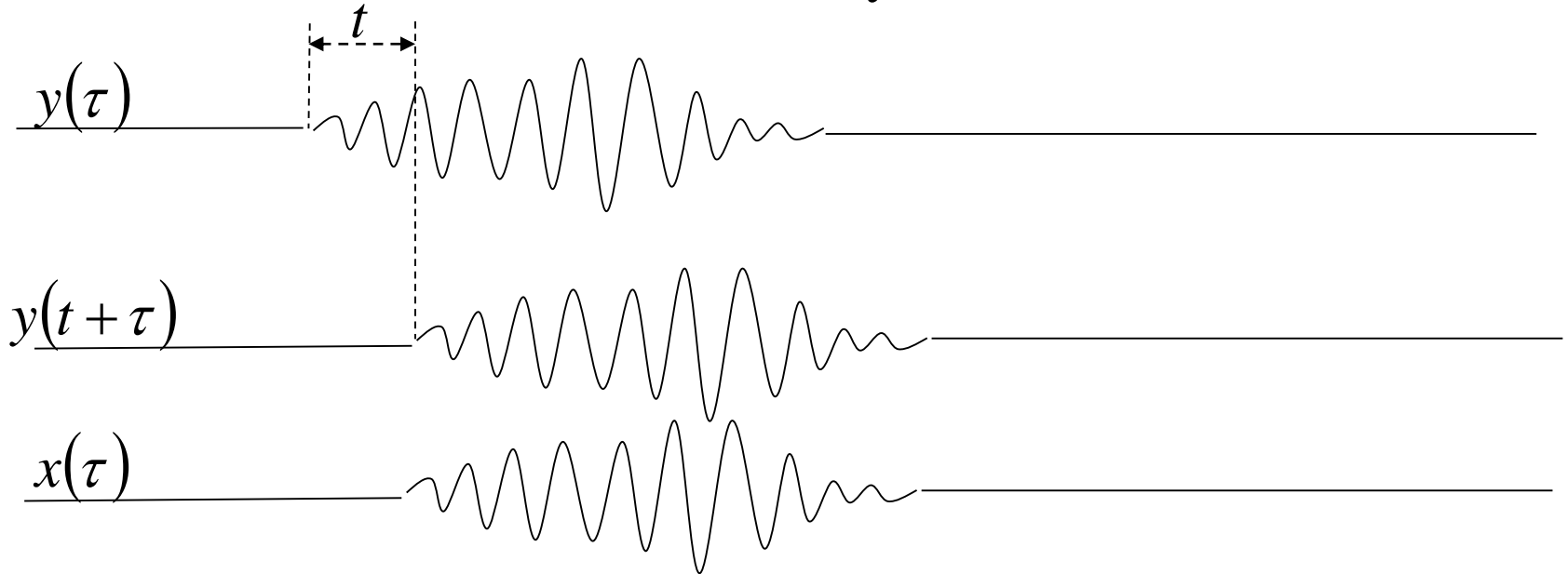
$$v = c + \frac{dv}{dk} \quad k = \frac{\omega}{c}$$

Dependency of Phase Velocity on the Frequency f (or Wave Number k): Dispersion.
 c is different from v in Dispersive Media.

Preparation: Phase Velocity Determination

Definition of Cross-correlation

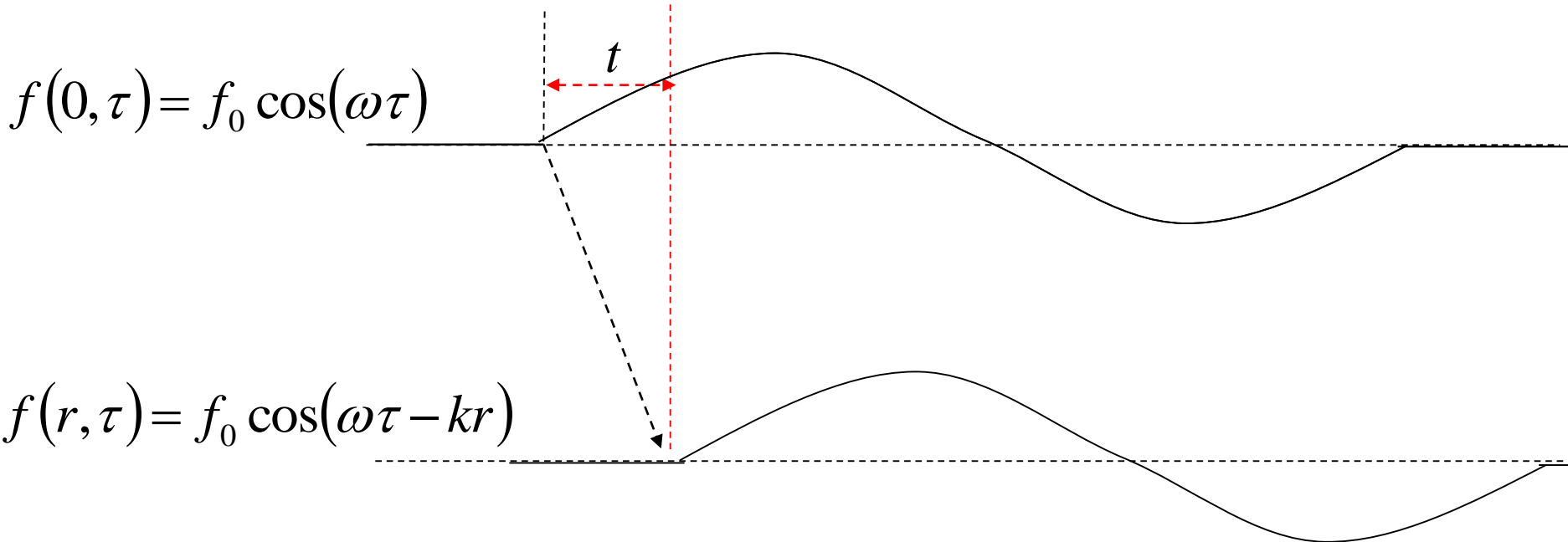
$$C_{xy}(t) \equiv \int x(\tau)y(t + \tau)d\tau$$



Product & Integration \longrightarrow **Cross-correlation**

Preparation: Phase Velocity Determination by Cross-Correlation

Cosine Function with the angular frequency ω (one cycle only)



Time lag t gives coincidence. If distance is r , the phase velocity c is given by r/t .

$$\begin{aligned} Cc(0, r, t) &= f_0^2 \int \cos(\omega \tau) \cos(\omega \tau + \omega t - kr) d\tau \\ &= f_0^2 \left[\int \cos^2(\omega \tau) d\tau \cos(\omega t - kr) - \int \cos(\omega \tau) \sin(\omega \tau) d\tau \sin(\omega t - kr) \right] \\ &\propto f_0^2 \cos(\omega t - kr) \quad \text{The maximum value of } Cc \text{ corresponds to this time lag.} \end{aligned}$$

In the frequency domain:

$$Cc(0, r, \omega) = F(0, \omega) \cdot \overline{F(r, \omega)} = |F(0, \omega)| \cdot |F(r, \omega)| \cdot \exp(i\Delta\phi(\omega))$$

Phase lag due to wave propagation is

$$\Delta\phi = \frac{\omega r}{c} \quad \text{because} \quad \exp\left\{i\omega\left(t + \frac{r}{c}\right)\right\} = \exp\left\{i\left(\omega t + \frac{\omega r}{c}\right)\right\}$$

Therefore,

$$Cc(0, r, \omega) = |F(0, \omega)| \cdot |F(r, \omega)| \cdot \exp\left(i\frac{\omega r}{c}\right)$$

Coherence

$$Coh(0, r, \omega) = \operatorname{Re}\left[\frac{Cc(0, r, \omega)}{|F(0, \omega)| \cdot |F(r, \omega)|}\right] = \operatorname{Re}\left[\exp\left(i\frac{\omega r}{c}\right)\right] = \cos\left(\frac{\omega r}{c}\right)$$

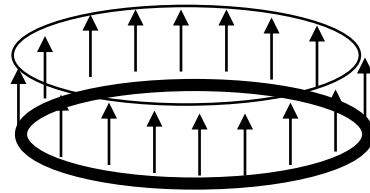
Here, c is the phase velocity measured along the measurement line.

Auto-Correlation

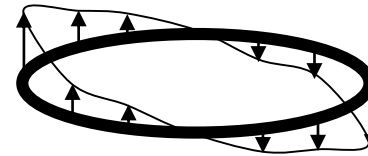
$$Ac(0, \omega) = Cc(0, 0, \omega) = |F(0, \omega)|^2, \quad Ac(r, \omega) = Cc(r, r, \omega) = |F(r, \omega)|^2$$

What's CCA ?

Zero and 1st order Fourier transforms of the wave field along the circle over the azimuth.



Zero Order



1st Order

CCA (Cho et al., 2006, Tada et al., 2007)

$$s_{CCA}(r, \omega) = \frac{PSD \left\langle \int_{-\pi}^{\pi} Z(t, r, \theta) d\theta \right\rangle}{PSD \left\langle \int_{-\pi}^{\pi} Z(t, r, \theta) \exp(-i\theta) d\theta \right\rangle} = \frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$

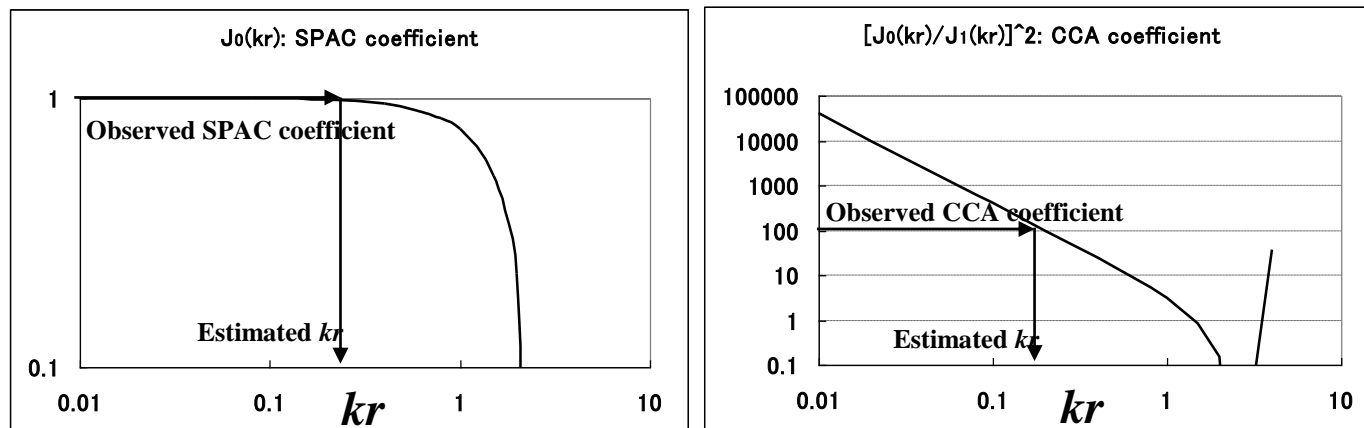
PSD<> denotes the power spectral density

SPAC (Aki, 1957;1965; Okada et al. 1987; Okada, 2003)

$$\rho(r, \omega) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\text{Re}\{E[C_{c,m}(r, \omega)]\}}{\sqrt{E[C_{c,c}(\omega)]E[C_{m,m}(\omega)]}} d\theta = J_0(kr)$$

CCA (Cho et al., 2006, Tada et al., 2007)

$$s_{CCA}(r, \omega) = \frac{\text{PSD}\left\langle \int_{-\pi}^{\pi} Z(t, r, \theta) d\theta \right\rangle}{\text{PSD}\left\langle \int_{-\pi}^{\pi} Z(t, r, \theta) \exp(-i\theta) d\theta \right\rangle} = \frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$



Schematic graphs for comparison of SPAC with CCA.

Analysis in the frequency domain

$$\begin{aligned}
 G_{Z_0Z_0}(r, r, \omega) &= PSD \left\langle \int_{-\pi}^{\pi} Z(t, r, \theta) d\theta \right\rangle \\
 &\approx PSD \left\langle \left\{ \frac{2\pi}{M} \sum_{m=1}^M Z(\omega, r, \theta_m) \right\} \left\{ \frac{2\pi}{M} \sum_{m'=1}^M Z(\omega, r, \theta_{m'})^* \right\} \right\rangle = \frac{4\pi^2}{M^2} \sum_{m=1}^M \sum_{m'=1}^M E[C_{m,m'}(r, \omega)]
 \end{aligned}$$

$$\begin{aligned}
 G_{Z_1Z_1}(r, r, \omega) &= PSD \left\langle \int_{-\pi}^{\pi} Z(t, r, \theta) \exp(-i\theta) d\theta \right\rangle \\
 &\approx PSD \left\langle \left\{ \frac{2\pi}{M} \sum_{m=1}^M Z(\omega, r, \theta_m) \exp(-i\theta_m) \right\} \left\{ \frac{2\pi}{M} \sum_{m'=1}^M Z(\omega, r, \theta_{m'}) \exp(-i\theta_{m'}) \right\}^* \right\rangle \\
 &= \frac{4\pi^2}{M^2} \sum_{m=1}^M \sum_{m'=1}^M E[C_{m,m'}(r, \omega)] \exp\{-i(\theta_m - \theta_{m'})\}
 \end{aligned}$$

Analysis in the frequency domain

CCA coefficient

$$s_{CCA'}(r, \omega) \approx \frac{\sum_{m=1}^M \sum_{m'=1}^M E[C_{m,m'}(r, \omega)]}{\sum_{m=1}^M \sum_{m'=1}^M E[C_{m,m'}(r, \omega)] \exp\{-i(\theta_m - \theta_{m'})\}} \approx \frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$

Analysis in the frequency domain

$$\begin{aligned}
 G_{Z_0Z_1}(r, r, \omega) &= \text{CSD} \left\langle \int_{-\pi}^{\pi} Z(t, r, \theta) d\theta, \int_{-\pi}^{\pi} Z(t, r, \theta) \exp(-i\theta) d\theta \right\rangle \\
 &\approx E \left[\left\{ \frac{2\pi}{M} \sum_{m'=1}^M Z(\omega, r, \theta_{m'}) \right\} \left\{ \frac{2\pi}{M} \sum_{m=1}^M Z(\omega, r, \theta_m) \exp(-i\theta_m) \right\}^* \right] \\
 &= \frac{4\pi^2}{M^2} \sum_{m=1}^M \sum_{m'=1}^M E[C_{m,m'}(r, \omega)] \exp(-i\theta_m)
 \end{aligned}$$

Incoming azimuth

$$\phi_R = \text{Arg}[G_{Z_0Z_1}(r, r, \omega)] - \frac{\pi}{2}$$

$$\begin{aligned}
G_{Z_0Z_0}(0, r, \omega) &= \text{CSD} \left\langle \int_{-\pi}^{\pi} Z(t, 0, 0) d\theta, \int_{-\pi}^{\pi} Z(t, r, \theta) d\theta \right\rangle \\
&= 2\pi E \left[Z(\omega, 0, 0) \left\{ \int_{-\pi}^{\pi} Z(\omega, r, \theta) d\theta \right\}^* \right] = 2\pi E \left[\int_{-\pi}^{\pi} Z(\omega, 0, 0) Z(\omega, r, \theta)^* d\theta \right] \\
&= 2\pi E \left[\int_{-\pi}^{\pi} C_{0,\theta}(r, \omega) d\theta \right] \approx \frac{4\pi^2}{M} \sum_{m=1}^M E[C_{0,m}(r, \omega)]
\end{aligned}$$

$$\begin{aligned}
G_{Z_0Z_0}(0, 0, \omega) &= \text{CSD} \left\langle \int_{-\pi}^{\pi} Z(t, 0, 0) d\theta, \int_{-\pi}^{\pi} Z(t, 0, 0) d\theta \right\rangle \\
&= 4\pi^2 E \left[Z(\omega, 0, 0) Z(\omega, 0, 0)^* \right] = 4\pi^2 E[C_{0,0}(r, \omega)]
\end{aligned}$$

SPAC coefficient

$$\begin{aligned}
\rho(r, \omega) &= \frac{1}{2\pi} \frac{\int_{-\pi}^{\pi} E[C_{0,\theta}(r, \omega)] d\theta}{E[C_{0,0}(r, \omega)]} = \frac{G_{Z_0Z_0}(0, r, \omega)}{G_{Z_0Z_0}(0, 0, \omega)} \\
&\approx \frac{1}{M} \frac{\sum_{m=1}^M E[C_{0,m}(r, \omega)]}{E[C_{0,0}(0, \omega)]} = J_0(r\omega/c)
\end{aligned}$$

Correction for incoherent noise (Cho et al. 2006)

$$s_{CCA}(r, \omega) = \frac{J_0^2(r\omega/c) + \varepsilon(\omega)/N}{J_1^2(r\omega/c) + \varepsilon(\omega)/N}$$

$$coh^2(\omega) = \frac{|G_{z_0z_0}(0, r, \omega)|^2}{G_{z_0z_0}(r, r, \omega)G_{z_0z_0}(0, 0, \omega)},$$

$$A(\omega) = -\rho^2(\omega)^2,$$

$$B(\omega) = \frac{\rho^2(\omega)}{coh^2(\omega)} - 2\rho^2(\omega) - \frac{1}{N},$$

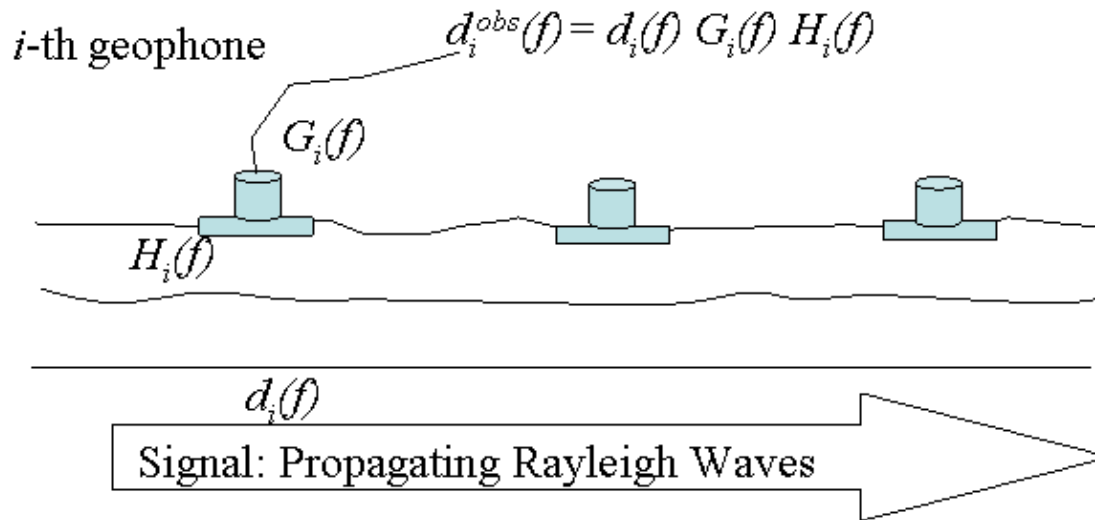
$$C(\omega) = \rho^2(\omega) \left\{ \frac{1}{coh^2(\omega)} - 1 \right\},$$

$$\varepsilon(\omega) = \left\{ -B(\omega) - \sqrt{B^2(\omega) - 4A(\omega)C(\omega)} \right\} / 2A(\omega).$$

System Correction for CCA in the frequency domain.

Obstacle: Difference of System Characteristics
& Very Local Amplification

Yokoi (2012)



Record of i -th channel Ground motion of propagating waves

$$d_i^{obs}(f) = d_i(f) \cdot G_i(f) \cdot H_i(f)$$

System Characteristics

Very Local Effect

$$G_i(f) = |G_i(f)| \exp\{-j\phi_i\}$$

$$H_i(f) = |H_i(f)| \exp\{-j\phi_i\}$$

Cross spectra between i -th and k -th channels' records

$$\begin{aligned} C_{ik}^{obs}(f) &= d_i^{obs}(f) \cdot d_k^{obs}(f)^* \\ &= C_{ik}(f) \cdot G_i(f) \cdot G_k(f)^* \cdot H_i(f) \cdot H_k(f)^* \end{aligned}$$

↙ Cross spectra between i -th and k -th channels' ground motion

CCA coefficient is defined using cross spectra of ground motion

$$s_{CCA}(r, \omega) = \frac{\sum_{i=1}^M \sum_{k=1}^M E[C_{ik}(\omega)]}{\sum_{i=1}^M \sum_{k=1}^M E[C_{ik}(\omega)] \exp\{-j(\theta_i - \theta_k)\}}$$

Phase difference among the system characteristics

$$\begin{aligned}
 \exp\{-j(\varphi_i - \varphi_k)\} &= \frac{G_i(\omega) \cdot G_k(\omega)^*}{|G_i(\omega)||G_k(\omega)|} = \frac{|d^{huddle}(\omega)|^2 G_i(\omega) \cdot G_k(\omega)^*}{|d^{huddle}(\omega)|^2 |G_i(\omega)||G_k(\omega)|} \\
 &= \frac{|d^{huddle}(\omega)|^2 G_i(\omega) \cdot G_k(\omega)^*}{\sqrt{|d^{huddle}(\omega)|^2 |G_i(\omega)|^2 |d^{huddle}(\omega)|^2 |G_k(\omega)|^2}} \\
 &= \frac{C_{ik}^{huddle}(\omega)}{\sqrt{C_{ii}^{huddle}(\omega) \cdot C_{kk}^{huddle}(\omega)}}
 \end{aligned}$$

Correction factor for phase difference among the system characteristics

$$Cor_{ik}^{huddle}(\omega) = \exp\{j(\varphi_i - \varphi_k)\} = \frac{\sqrt{C_{ii}^{huddle}(\omega) \cdot C_{kk}^{huddle}(\omega)}}{C_{ik}^{huddle}(\omega)}$$

For a stable processing, average over time blocks is applied

$$\overline{C_{ik}^{huddle}(f)} = \exp \left\{ jE \left[\text{Arg} \left(\frac{\sqrt{C_{ii}^{huddle}(f)} \cdot C_{kk}^{huddle}(f)}}{C_{ik}^{huddle}(f)} \right) \right] \right\}$$

Corrected cross spectra

$$\begin{aligned} C_{ik}^{cor}(\omega) &= C_{ik}^{obs}(\omega) \cdot \overline{C_{ik}^{huddle}(\omega)} \\ &= C_{ik}(\omega) |G_i(\omega)| |G_k(\omega)| H_i(\omega) H_k(\omega)^* \end{aligned}$$

Complex coherence between i -th and k -th channels' corrected records

$$Coh_{ik}^{cor}(\omega) = \frac{E[C_{ik}^{cor}(\omega)]}{\sqrt{E[C_{ii}^{cor}(\omega)]E[C_{kk}^{cor}(\omega)]}} = \frac{E[C_{ik}(\omega)]}{\sqrt{E[C_{ii}(\omega)]E[C_{kk}(\omega)]}} \cdot \exp\{-j(\phi_i - \phi_k)\}$$

An interim quantity $R_{ik}(\omega)$

$$R_{ik}(\omega) = C_{00}^{cor}(\omega) \cdot Coh_{ik}^{cor}(\omega) = C_{00}^{cor}(\omega) \cdot \frac{E[C_{ik}^{cor}(\omega)]}{\sqrt{E[C_{ii}^{cor}(\omega)]E[C_{kk}^{cor}(\omega)]}}$$

$$= C_{00}^{obs}(\omega) \cdot \frac{E[C_{ik}^{obs}(\omega)]Cor_{ik}^{huddle}(\omega)}{\sqrt{E[C_{ii}^{obs}(\omega)]Cor_{ii}^{huddle}(\omega)E[C_{kk}^{obs}(\omega)]Cor_{kk}^{huddle}(\omega)}}$$

$$= \frac{C_{00}^{obs}(\omega)E[C_{ik}^{obs}(\omega)]Cor_{ik}^{huddle}(\omega)}{\sqrt{E[C_{ii}^{obs}(\omega)]E[C_{kk}^{obs}(\omega)]}}$$

:Description using observed records

On the other hand,

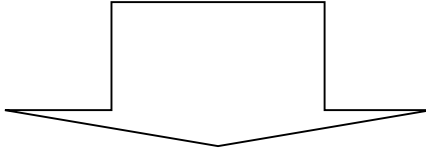
$$R_{ik}(\omega) = C_{00}^{obs}(\omega) \cdot \frac{E[C_{ik}(\omega)]}{\sqrt{E[C_{ii}(\omega)]E[C_{kk}(\omega)]}} \cdot \exp\{-j(\phi_i - \phi_k)\}$$

Assumptions:

1) Phase difference due to very local effect is negligible $\phi_i - \phi_k \approx 0$

2) Power spectra of ground motion is same for all channels

$$P(\omega) = E[C_{ii}(\omega)] = E[C_{kk}(\omega)]$$



$$R_{ik}(\omega) \approx \left(\frac{C_{00}^{obs}(\omega)}{P(\omega)} \right) E[C_{ik}(\omega)]$$

Contents of () is common for all channel pairs, then

$$s_{CCA}(\omega) = \frac{\sum_{i=1}^M \sum_{k=1}^M E[C_{ik}(\omega)]}{\sum_{i=1}^M \sum_{k=1}^M E[C_{ik}(\omega)] \exp\{-j(\theta_i - \theta_k)\}} \approx \frac{\sum_{i=1}^M \sum_{k=1}^M R_{ik}(\omega)}{\sum_{i=1}^M \sum_{k=1}^M R_{ik}(\omega) \exp\{-j(\theta_i - \theta_k)\}}$$

Complete definition of CCA coefficient is obtained by using $R_{ik}(\omega)$ in place of $C_{ik}^{obs}(\omega)$

SPAC coefficient (Aki 1957, Okada 2003)

$$\rho(r, \omega) \approx \frac{1}{M} \frac{\sum_{m=1}^M E[C_{0,m}(r, \omega)]}{E[C_{0,0}(0, \omega)]} = J_0(r\omega/c)$$

Use $R_{0,m}(\omega)$ in place of $E[C_{0,m}(\omega)]$ $R_{ik}(\omega) = \frac{C_{00}^{obs}(\omega) E[C_{ik}^{obs}(\omega)] \overline{Cor_{ik}^{huddle}(\omega)}}{\sqrt{E[C_{ii}^{obs}(\omega)] E[C_{kk}^{obs}(\omega)]}}$

Neglect phase difference of system characteristic

$$\overline{Cor_{ik}^{huddle}(\omega)} = \exp\{jE[\varphi_i - \varphi_k]\} \approx 1$$

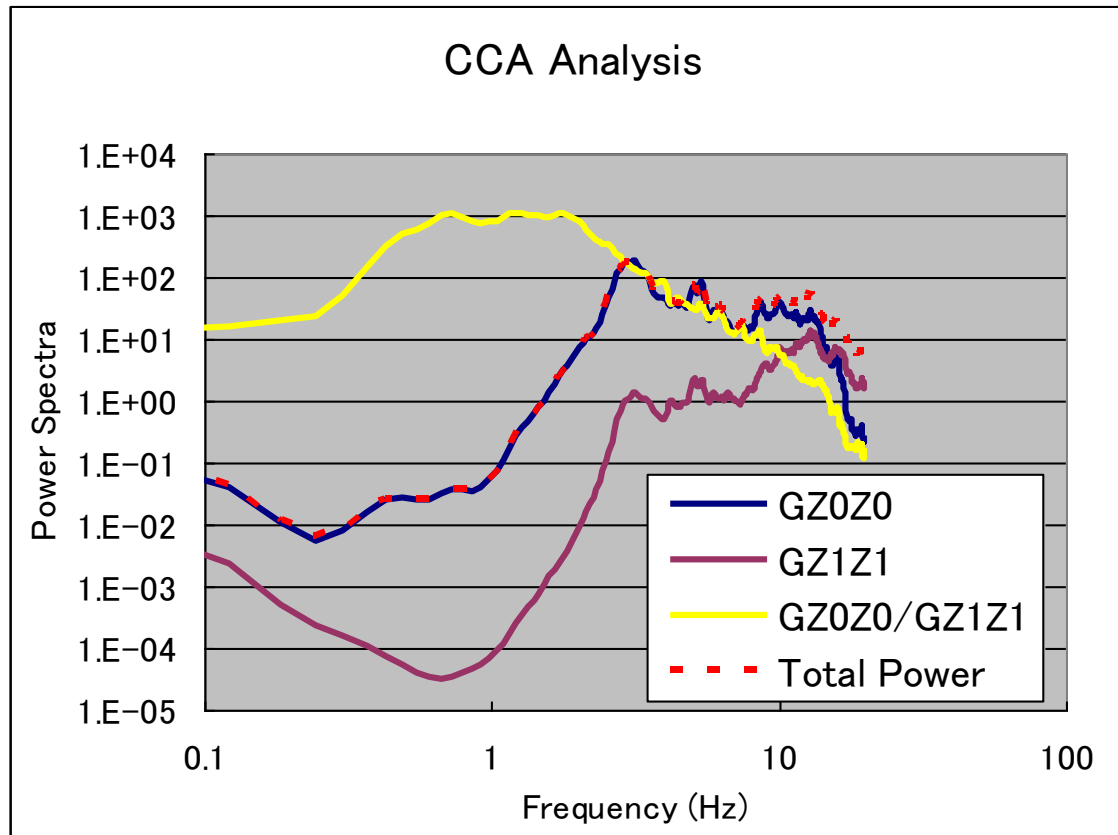
Or perform system correction

$$E[C_{ik}^{obs}(\omega)] \overline{Cor_{ik}^{huddle}(\omega)} \Rightarrow E[C_{ik}^{obs}(\omega)]$$

Okada et al. (1987)'s formula is derived by two assumptions in the previous slide

$$\rho(r, \omega) \approx \frac{1}{M} \frac{\sum_{m=1}^M E[C_{0,m}(r, \omega)]}{E[C_{0,0}(0, \omega)]} = \frac{1}{M} \frac{\sum_{m=1}^M R_{0,m}(\omega)}{R_{0,0}(\omega)} \approx \frac{1}{M} \sum_{m=1}^M \frac{E[C_{0,m}^{obs}(\omega)]}{\sqrt{E[C_{0,0}^{obs}(\omega)] E[C_{m,m}^{obs}(\omega)]}}$$

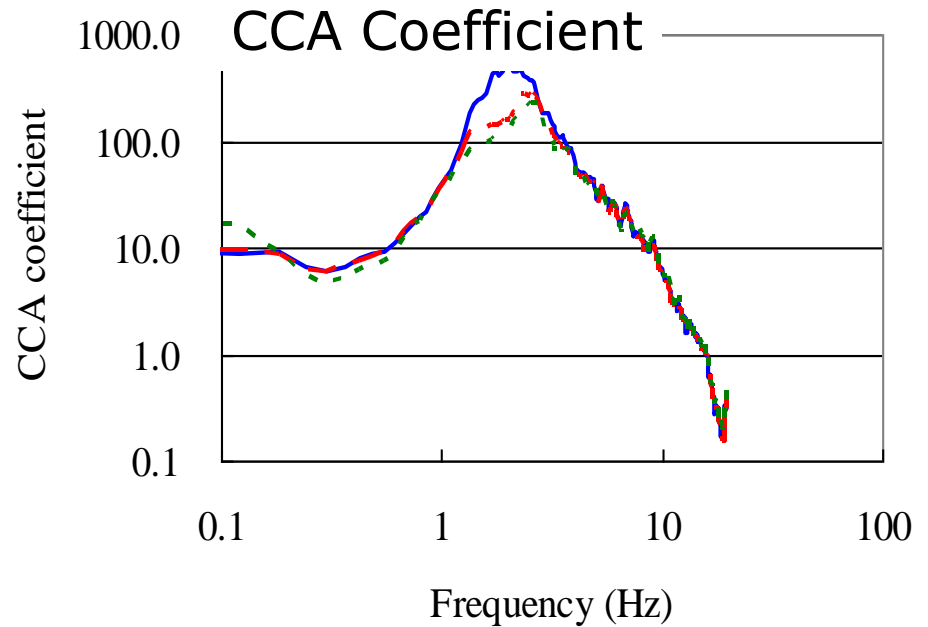
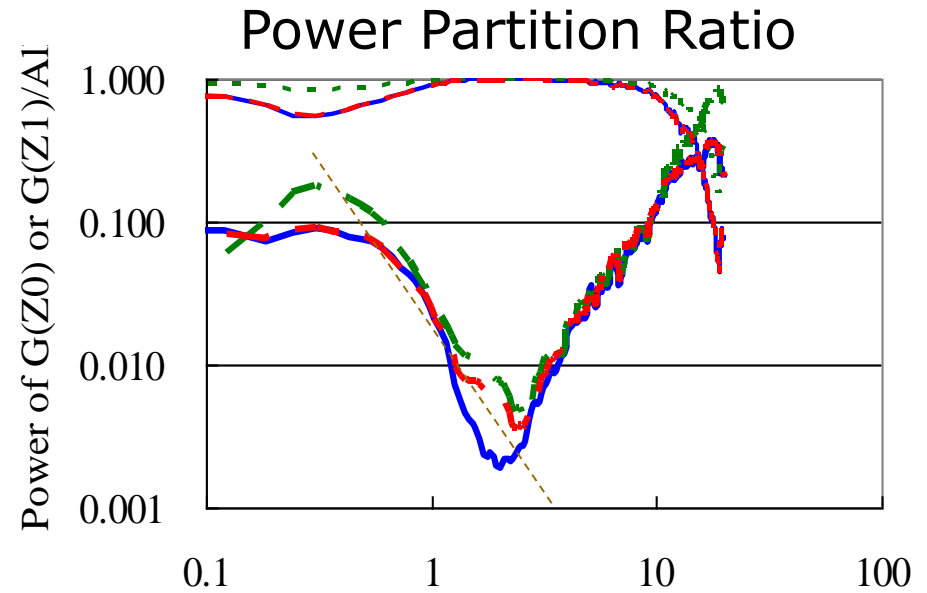
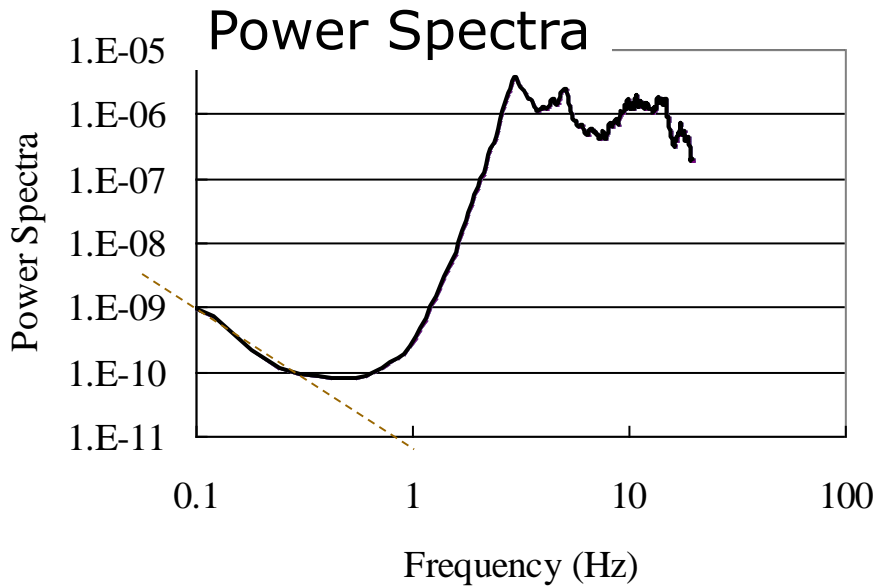
$$s_{CCA}(r, \omega) = \frac{G_{Z_0Z_0}(r, r, \omega)}{G_{Z_1Z_1}(r, r, \omega)} = \frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$



A real example of CCA Analysis
(BRI, Tsukuba, Japan)

CCA

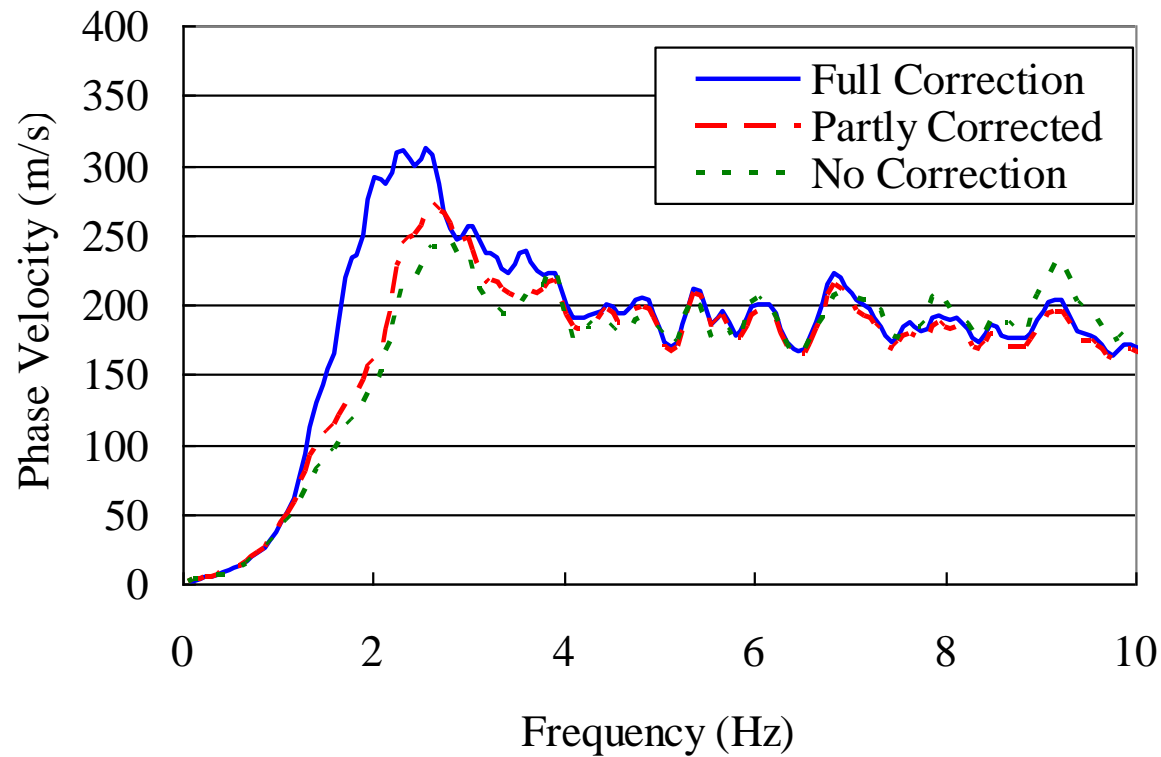
Example using 2Hz
seismometers



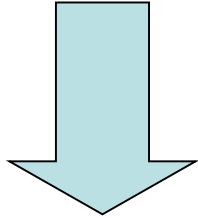
System Correction

Phase velocity

Effect of System Correction

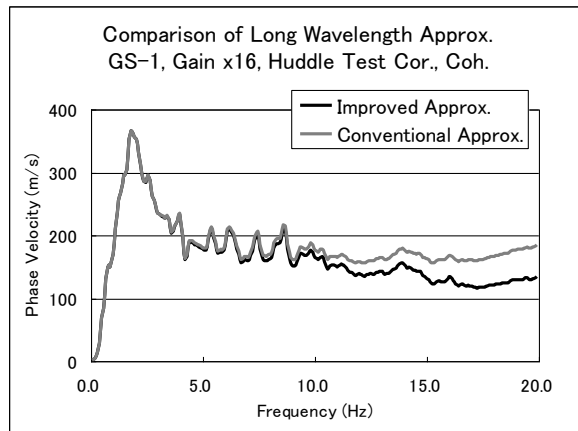


$$s(r, \omega) = \frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$

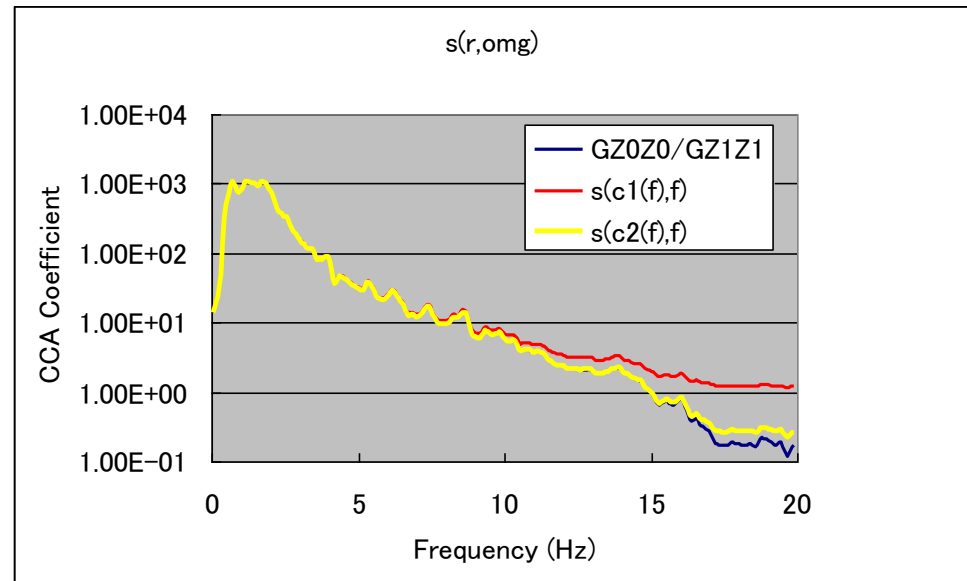


Long wavelength approximation
(Small value of kr) (Cho et al, 2006)

$$c(\omega) = \frac{r\omega}{2} \sqrt{s(r, \omega) + 2}$$



c1

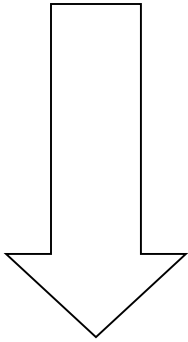


Improved approximation:
Yokoi(2012)

c2

$$c(\omega)/r\omega = 1.0003 \left(\frac{\sqrt{s(r, \omega) + 0.97221}}{2.0003} + 0.0015245 \right) - 0.0138$$

Dispersion Curve



Heuristic Search: VFSA-DHSM

Vs Structure