Dynamic Soil Structure Interaction

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One-Dimensional Shear Wave Propagation in Two-Layered Strata

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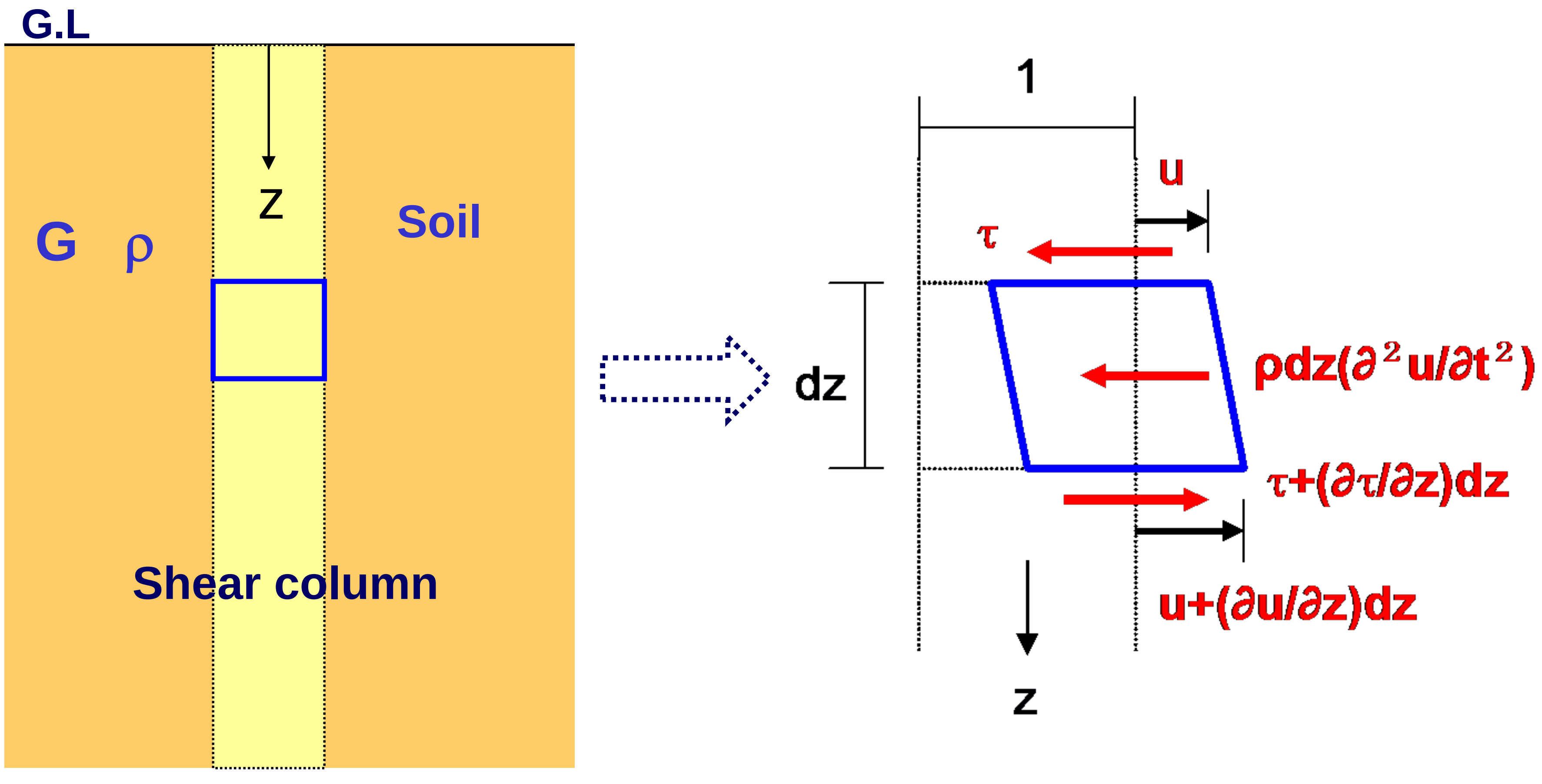
One-Dimensional Shear Wave Propagation Theory

Consider a shear column with unit cross-section area, and it has *G* of shear modulus and ρ of mass density.



(1) Eq. of Motion

$$-\rho \frac{\partial^2 u}{\partial t^2} dz + (\tau + \frac{\partial \tau}{\partial z} dz) - \tau = 0 \qquad (1)$$

$$\therefore \quad \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau}{\partial z} \qquad (2)$$

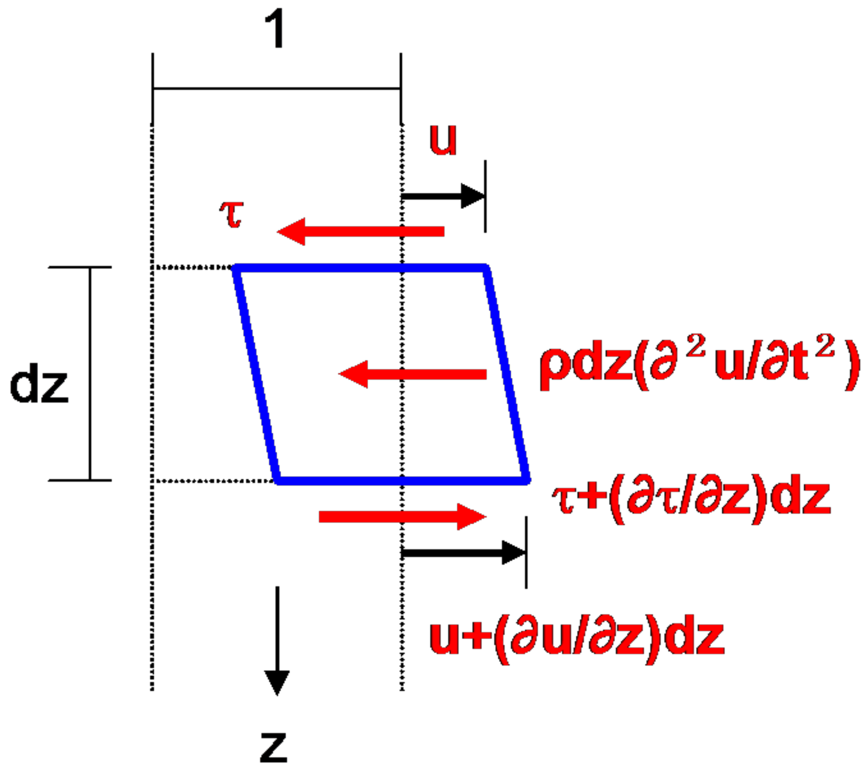


Fig-1 Shear Column

Shear stress τ is given by:

$$\tau = G \frac{\partial u}{\partial z} \qquad (3)$$

Putting Eq.(3) into Eq.(2):

$$\rho \frac{\partial^2 u}{\partial t^2} = G \frac{\partial^2 u}{\partial z^2} \qquad (4)$$

Eq.(4) is an equation of motion for one-dimensional shear wave propagation.

(2) Solution of Eq.(4)

The velocity *V* of shear wave (S-wave) is expressed by:

$$V = \sqrt{G/\rho} \qquad (5)$$

Using *V*, Eq.(4) can be transferred to:

$$\frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial z^2} \qquad (6)$$

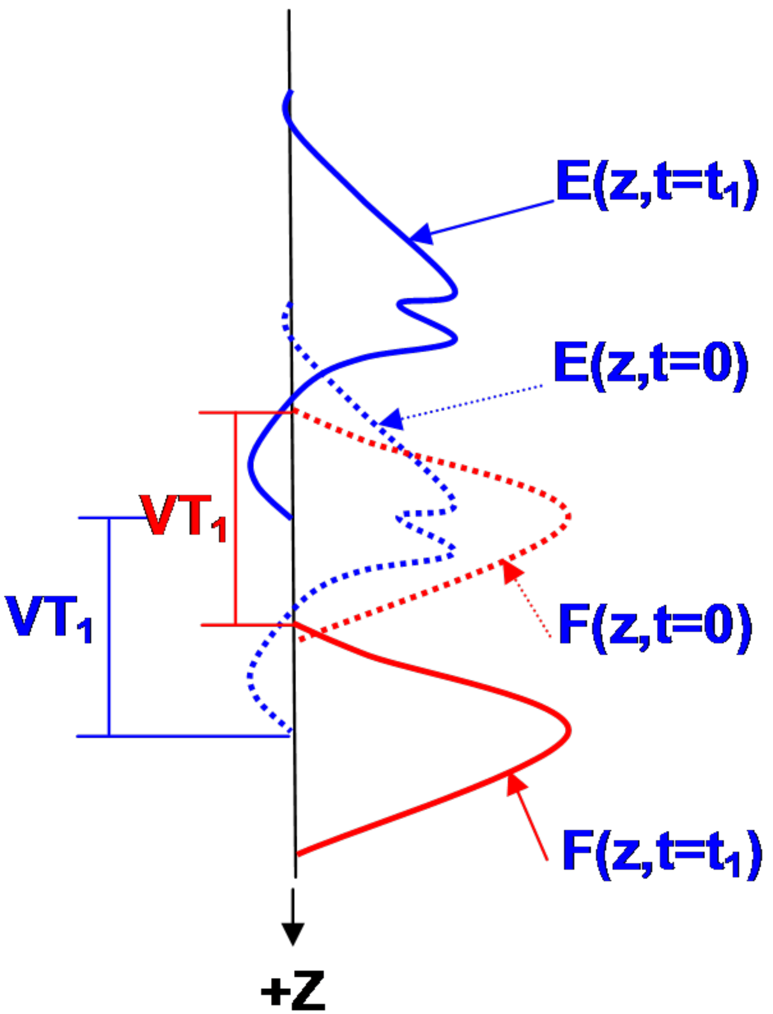
The solution of Eq.(6) is given by:

$$u(z,t) = E(t + \frac{z}{V}) + F(t - \frac{z}{V}) \qquad (7)$$

where, E(*t*+*z*/*V*) and F(*t*-*z*/*V*) are arbitrary functions.

E($t+z/V$) indicates the displacement due to a backward propagating wave which propagates in the negative z direction.

while F($t-z/V$) due to a forward propagating wave propagating in the positive z direction.



Wave propagation from t=0 to t=t_1

Putting : $\varsigma = t + \dfrac{z}{V}$, $\eta = t - \dfrac{z}{V}$ (8)

then, $u(z,t) = E(\varsigma) + F(\eta)$ (9)

The shear stress τ is also expressed by:

$$\tau(z,t) = G\frac{\partial u(z,t)}{\partial z} = G\{\frac{\partial E(t+z/V)}{\partial z} + \frac{\partial F(t-z/V)}{\partial z}\}$$

$$= G\{\frac{\partial E(\varsigma)}{\partial \varsigma}\frac{\partial \varsigma}{\partial z} + \frac{\partial F(\eta)}{\partial \eta}\frac{\partial \eta}{\partial z}\} = G\frac{1}{V}\{\frac{\partial E(\varsigma)}{\partial \varsigma} - \frac{\partial F(\eta)}{\partial \eta}\} \qquad (10)$$

The coefficient *G/V* denotes impedance:

$$\frac{G}{V} = \frac{\rho V^2}{V} = \rho V$$ **: Impedance** (11)

Eq.(10) is expressed by:

$$\tau(z,t) = \rho V\{\frac{\partial E(\varsigma)}{\partial \varsigma} - \frac{\partial F(\eta)}{\partial \eta}\} \qquad (12)$$

[problem-1] Proof that Eq.(7) is the solution of Eq.(6).

(proof)

$$\frac{\partial u(z,t)}{\partial z} = \frac{\partial E(\varsigma)}{\partial \varsigma}\frac{\partial \varsigma}{\partial z} + \frac{\partial F(\eta)}{\partial \eta}\frac{\partial \eta}{\partial z} = \frac{1}{V}\{\frac{\partial E(\varsigma)}{\partial \varsigma} - \frac{\partial F(\eta)}{\partial \eta}\} \quad \text{(a)}$$

$$\frac{\partial^2 u(z,t)}{\partial z^2} = \frac{1}{V}\{\frac{\partial^2 E(\varsigma)}{\partial \varsigma^2}\frac{\partial \varsigma}{\partial z} - \frac{\partial^2 F(\eta)}{\partial \eta^2}\frac{\partial \eta}{\partial z}\} = \frac{1}{V^2}\{\frac{\partial^2 E(\varsigma)}{\partial \varsigma^2} + \frac{\partial^2 F(\eta)}{\partial \eta^2}\} \quad \text{(b)}$$

$$\frac{\partial u(z,t)}{\partial t} = \frac{\partial E(\varsigma)}{\partial \varsigma}\frac{\partial \varsigma}{\partial t} + \frac{\partial F(\eta)}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial E(\varsigma)}{\partial \varsigma} + \frac{\partial F(\eta)}{\partial \eta} \quad \text{(c)}$$

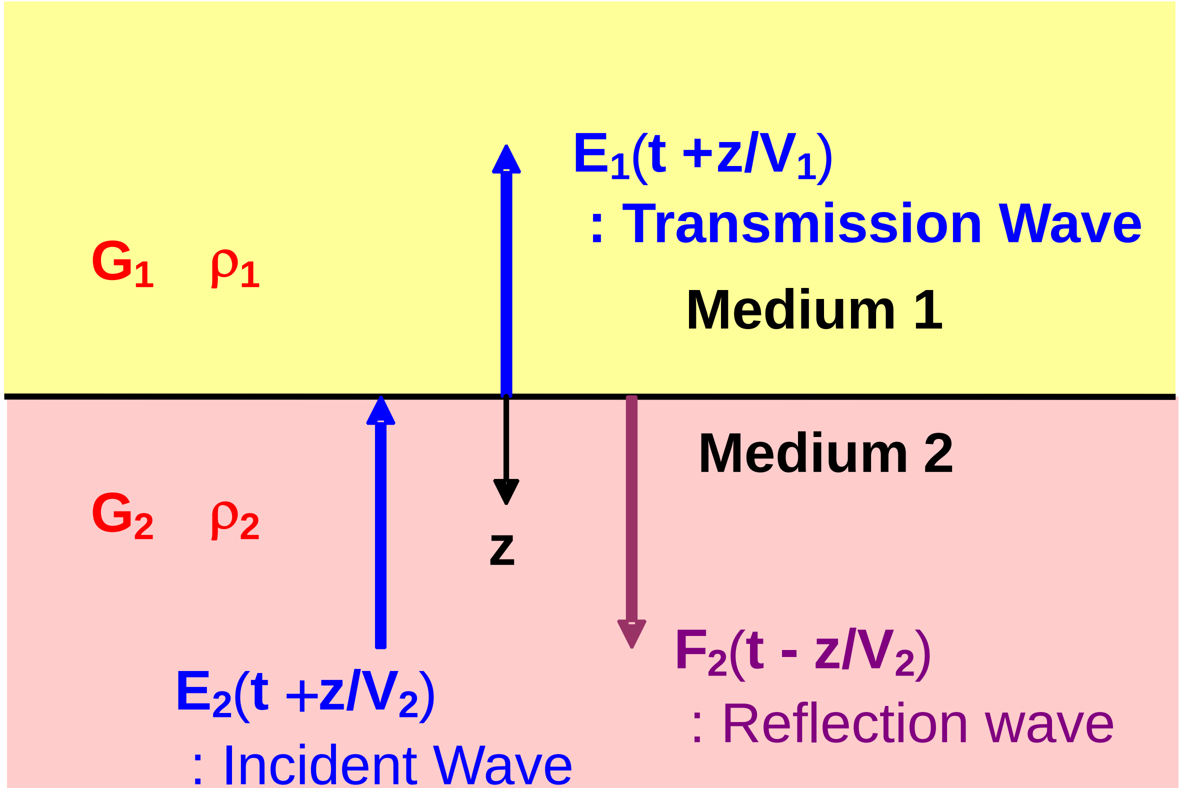
$$\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{\partial^2 E(\varsigma)}{\partial \varsigma^2}\frac{\partial \varsigma}{\partial t} + \frac{\partial^2 F(\eta)}{\partial \eta^2}\frac{\partial \eta}{\partial t} = \frac{\partial^2 E(\varsigma)}{\partial \varsigma^2} + \frac{\partial^2 F(\eta)}{\partial \eta^2} \quad \text{(d)}$$

Substituting Eq.(b) and Eq.(d) into Eq.(6):

$$\frac{\partial^2 u}{\partial t^2} - V^2\frac{\partial^2 u}{\partial z^2} = [\frac{\partial^2 E(\varsigma)}{\partial \varsigma^2} + \frac{\partial^2 F(\eta)}{\partial \eta^2}] - [V^2\frac{1}{V^2}\{\frac{\partial^2 E(\varsigma)}{\partial \varsigma^2} + \frac{\partial^2 F(\eta)}{\partial \eta^2}\}] = 0 \quad \text{(e)}$$

(3) Transmission and Reflection

Consider the wave propagation in two semi-infinite media as shown in the figure.



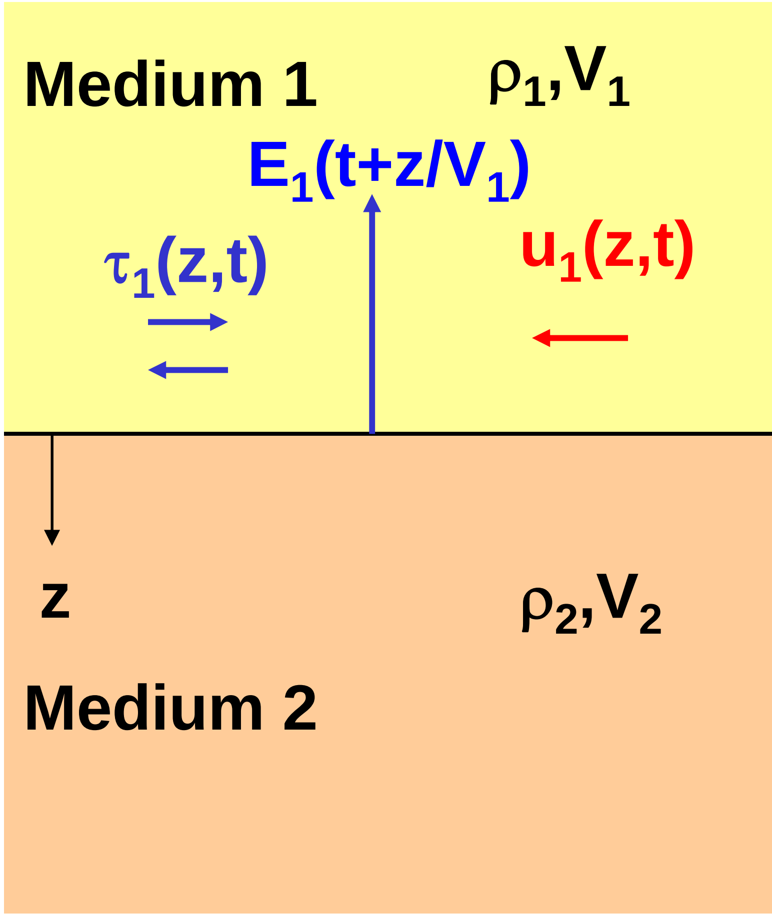
When the wave $E_2(t+z/V_2)$ propagates upward and reaches at the interface (z=0) between two media, this wave is divided into the transmission wave $E_1(t+z/V_1)$ and the reflection wave $F_2(t-z/V_2)$.

The displacement and shear stress in medium 1:

$$u_1(z,t) = E_1(t + z/V_1) = E_1(\varsigma_1) \qquad (13)$$

$$\tau_1(z,t) = G_1 \frac{\partial u_1(z,t)}{\partial z} = G_1 \frac{\partial E_1(\varsigma_1)}{\partial \varsigma_1}\frac{\partial \varsigma_1}{\partial z} = \frac{G_1}{V_1}\frac{\partial E_1(\varsigma_1)}{\partial \varsigma_1}$$

$$= \frac{\rho_1 V_1^2}{V_1}\frac{\partial E_1(\varsigma_1)}{\partial \varsigma_1} = \rho_1 V_1 \frac{\partial E_1(\varsigma_1)}{\partial \varsigma_1} \qquad (14)$$



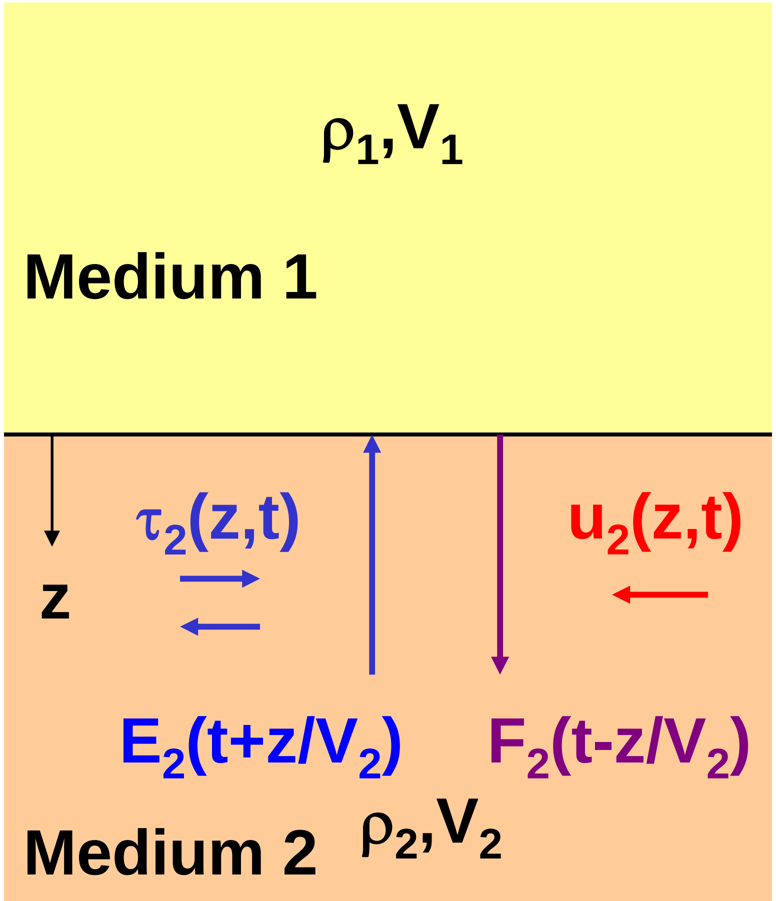
where,

$$\varsigma_1 = t + z/V_1 \qquad (15)$$

Similarly, for medium 2:

$$u_2(z,t) = E_2(t + z/V_2) + F_2(t - z/V_2) = E_2(\varsigma_2) + F_2(\eta_2) \qquad (16)$$

$$\tau_2(z,t) = \rho_2 V_2 \{\frac{\partial E_2(\varsigma_2)}{\partial \varsigma_2} - \frac{\partial F_2(\eta_2)}{\partial \eta_2}\} \qquad (17)$$



where,

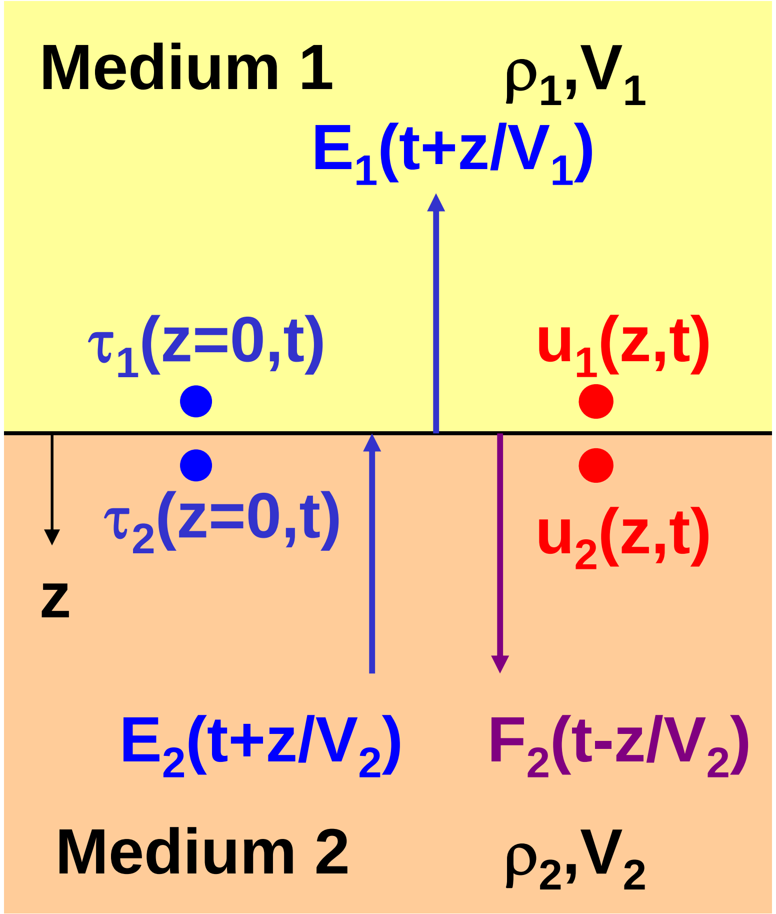
$$\varsigma_2 = t + z/V_2 \qquad (18)$$

$$\eta_2 = t - z/V_2 \qquad (19)$$

The boundary conditions at the interface(z=0):

$$u_1(z=0,t)=u_2(z=0,t) \qquad (20)$$

$$\tau_1(z=0,t)=\tau_2(z=0,t) \qquad (21)$$



Substituting Eq.(13): $u_1(z,t) = E_1(\varsigma_1)$

and Eq.(16) : $u_2(z,t) = E_2(\varsigma_2) + F_2(\eta_2)$

into Eq.(20): $u_1(z=0,t) = u_2(z=0,t)$

We obtain: $E_1(t) = E_2(t) + F_2(t)$ (22)

Similarly, putting Eq.(14): $\tau_1(z,t) = \rho_1 V_1 \dfrac{\partial E_1(\varsigma_1)}{\partial \varsigma_1}$

and Eq.(17) : $\tau_2(z,t) = \rho_2 V_2 \{\dfrac{\partial E_2(\varsigma_2)}{\partial \varsigma_2} - \dfrac{\partial F_2(\eta_2)}{\partial \eta_2}\}$

into Eq.(21): $\tau_1(z=0,t) = \tau_2(z=0,t)$

Then, $\rho_1 V_1 [\dfrac{\partial E_1(\varsigma_1)}{\partial \varsigma_1}]_{z=0} = \rho_2 V_2 [\dfrac{\partial E_2(\varsigma_2)}{\partial \varsigma_2} - \dfrac{\partial F_2(\eta_2)}{\partial \eta_2}]_{z=0}$

$$\therefore \quad \rho_1 V_1 \frac{\partial E_1(t)}{\partial t} = \rho_2 V_2 \{\frac{\partial E_2(t)}{\partial t} - \frac{\partial F_2(t)}{\partial t}\} \qquad (23)$$

Defining the impedance ratio α as:

$$\alpha = \frac{\rho_1 V_1}{\rho_2 V_2} \tag{24}$$

then, Eq.(23): $$\rho_1 V_1 \frac{\partial E_1(t)}{\partial t} = \rho_2 V_2 \left\{ \frac{\partial E_2(t)}{\partial t} - \frac{\partial F_2(t)}{\partial t} \right\}$$

leads to:

$$\alpha \frac{\partial E_1(t)}{\partial t} = \frac{\partial E_2(t)}{\partial t} - \frac{\partial F_2(t)}{\partial t} \tag{25}$$

Integrating Eq.(25) gives :

$$\alpha E_1(t) = E_2(t) - F_2(t) \tag{26}$$

Adding Eq.(22): $E_1(t) = E_2(t) + F_2(t)$

to Eq.(26): $\alpha E_1(t) = E_2(t) - F_2(t)$

then,

$$(1+\alpha)E_1(t) = 2E_2(t) \qquad (27)$$

$$\therefore \quad E_1(t) = \frac{2}{1+\alpha}E_2(t) \qquad (28)$$

Substituting Eq.(28) into Eq.(26): $\alpha E_1(t) = E_2(t) - F_2(t)$

then,

$$F_2(t) = E_2(t) - \alpha E_1(t) = (1 - \frac{2\alpha}{1+\alpha})E_2(t) = \frac{1-\alpha}{1+\alpha}E_2(t) \qquad (29)$$

Define the transmission *T* and reflection coefficient *R* as:

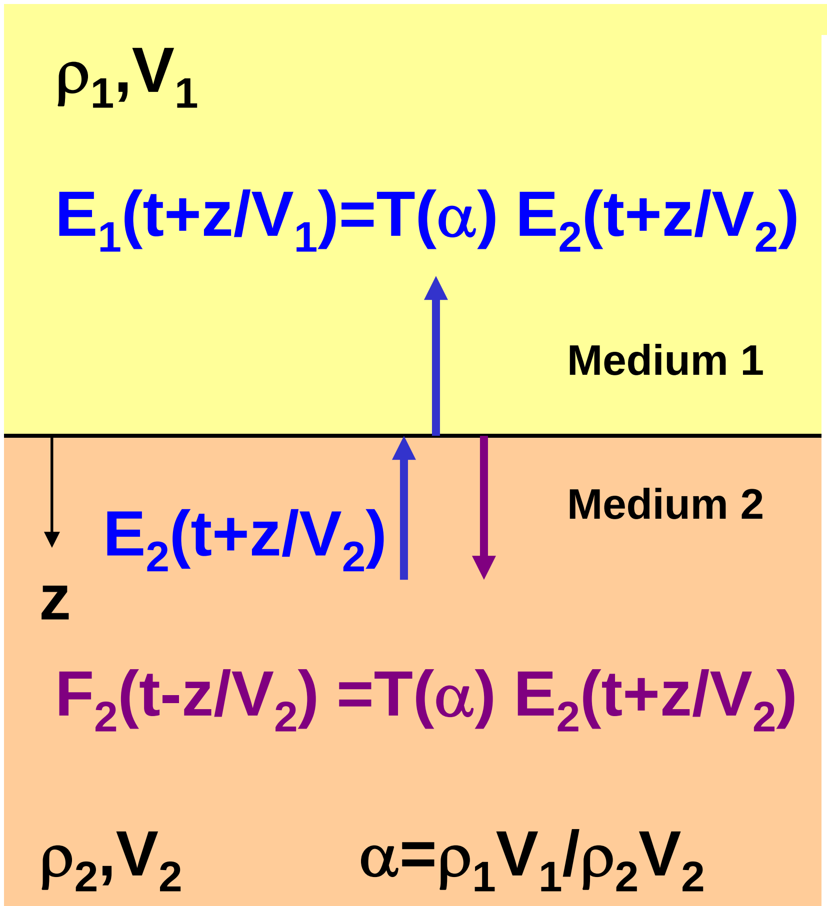
Transmission coefficient : $T = \frac{2}{1+\alpha}$ (30)

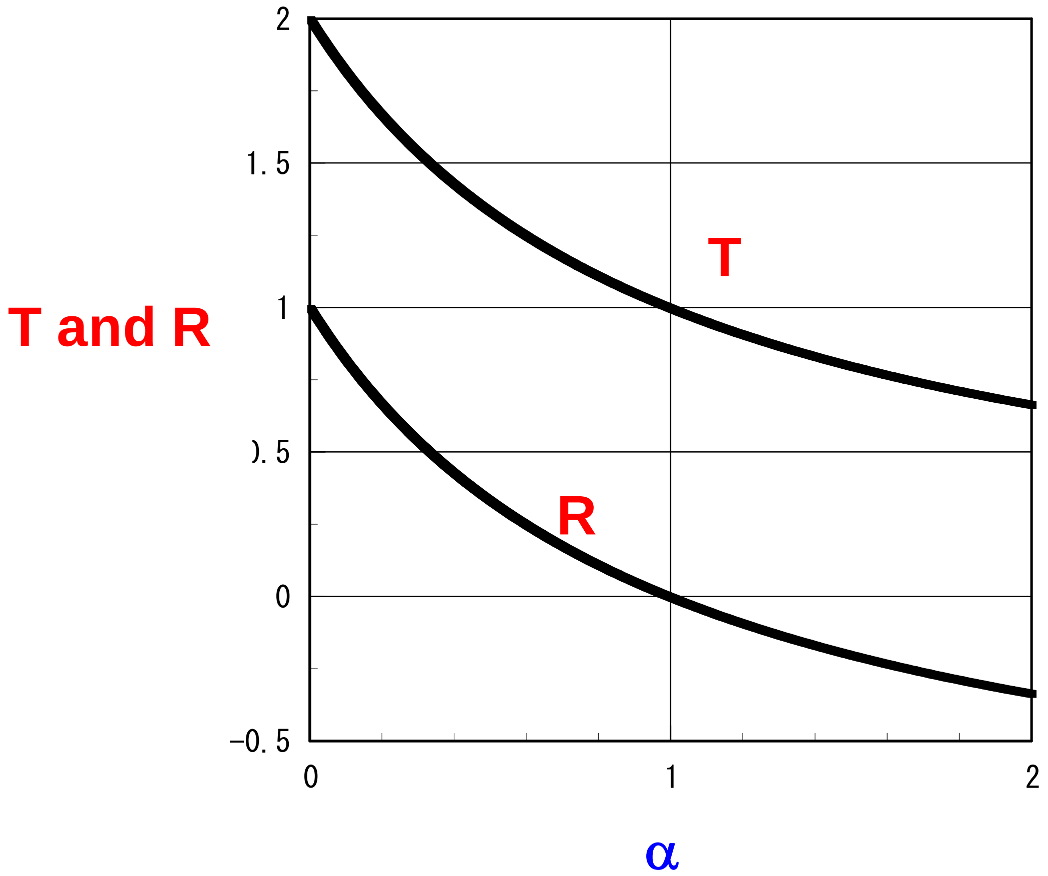
Reflection coefficient : $R = \frac{1-\alpha}{1+\alpha}$ (31)

Using *T* and *R*, the transmission wave $E_1(t+z/V_1)$ and the reflection wave $F_2(t-z/V_2)$ are given by:

$$E_1(t+z/V_1) = T(\alpha)E_2(t+z/V_2) \tag{32}$$

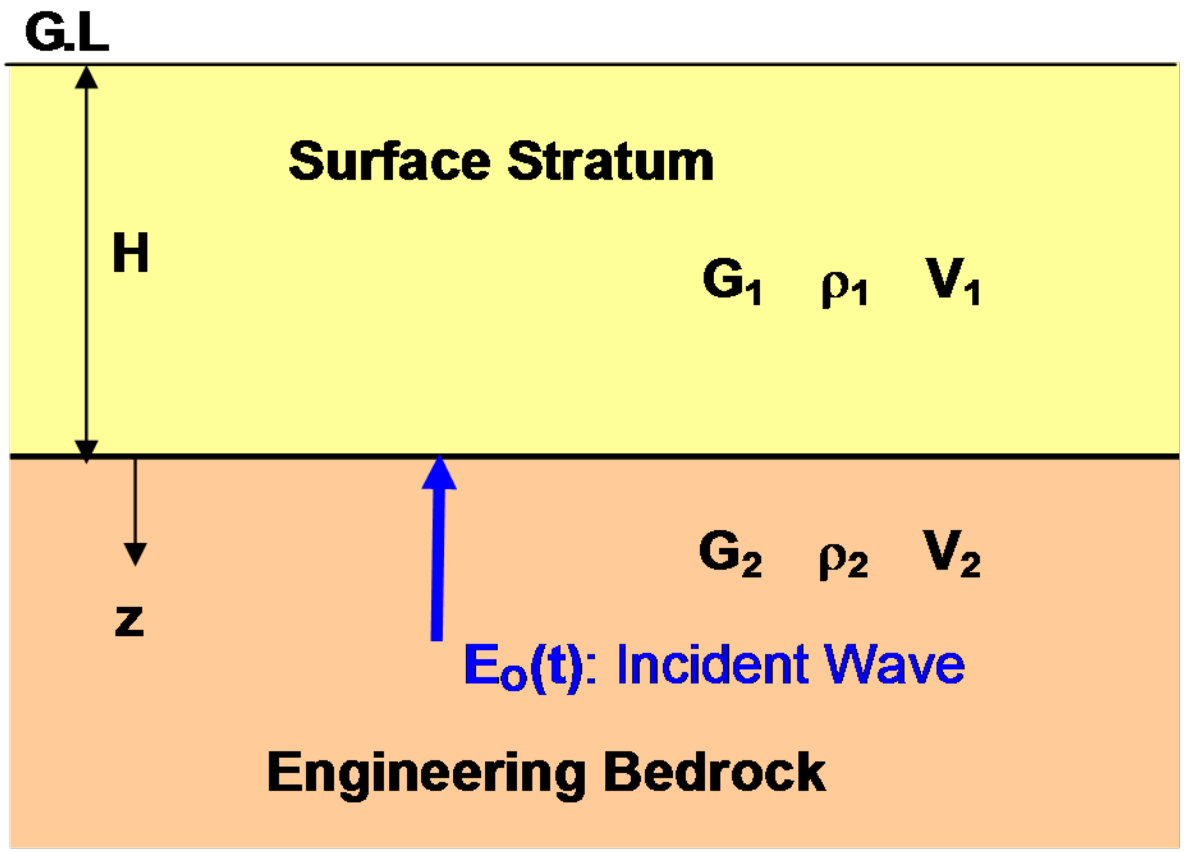
$$F_2(t-z/V_2) = R(\alpha)E_2(t+z/V_2) \tag{33}$$





(4) Amplification of a surface stratum on the engineering bedrock

We consider the SH wave propagation in a surface stratum on the engineering bedrock, when the SH wave $E_O(t)$ incidents.



The equation of the motion is given by Eq.(6):

$$\frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial z^2} \qquad (6)$$

Putting:

$$u(z,t) = U(z)e^{i\omega t} \qquad (34)$$

in which ω denotes circular frequency(rad./sec).

Substituting Eq.(34) into Eq.(6):

$$\frac{d^2U(z)}{dz^2} + \left(\frac{\omega}{V}\right)^2 U(z) = 0 \qquad (35)$$

$$\therefore \quad \frac{d^2U(z)}{dz^2} + \kappa^2 U(z) = 0 \qquad (36)$$

where, $\kappa = \frac{\omega}{V}$:**wave number**(rad.s/m) (37)

The solution of Eq.(36) : $$\frac{d^2U(z)}{dz^2} + \kappa^2 U(z) = 0$$

is given by:

$$U(z) = E \cdot e^{i\kappa z} + F \cdot e^{-i\kappa z} \qquad (38)$$

where *E* and *F* are arbitrary constants.

Substituting Eq.(38) into Eq.(34): $u(z,t) = U(z)e^{i\omega t}$

the displacement *u(z,t)* can be expressed by:

$$u(z,t) = E \cdot e^{i(\omega t + \kappa z)} + F \cdot e^{i(\omega t - \kappa z)} \qquad (39)$$

The first term indicates the wave propagating in the negative *z* direction, while the second term in the positive z direction. *E* and *F* express the amplitude of the waves.

Hereafter, the time term $e^{i\omega t}$ is not written.

The shear stress $\tau(z)$ is expressed by:

$$\tau(z) = G\frac{dU(z)}{dz} = G \cdot i\kappa(E \cdot e^{i\kappa z} - F \cdot e^{-i\kappa z}) \qquad (40)$$

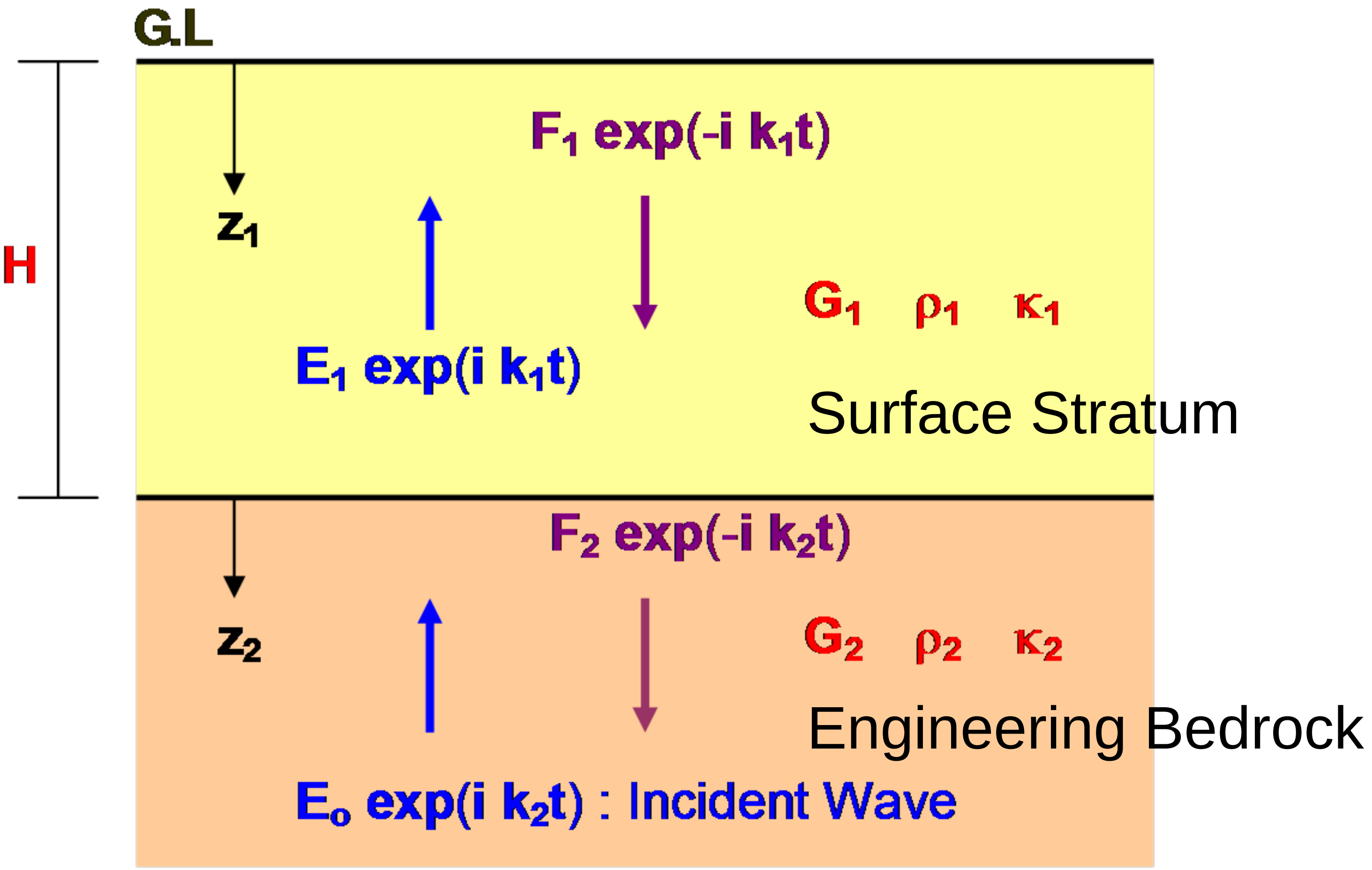
The coefficient of this equation is:

$$G\kappa = \rho V^2 \frac{\omega}{V} = (\rho V)\omega \qquad (41)$$

The expression of the shear stress $\tau(z)$ is transformed into:

$$\tau(z) = i(\rho V)\omega(E \cdot e^{i\kappa z} - F \cdot e^{-i\kappa z}) \qquad (42)$$

Consider the wave propagation in two layered strata as shown, when the SH wave $E_o \exp(i\kappa_2 t)$ incidents on the interface between the surface stratum and the engineering bedrock.



The displacement and shear stress in both strata are given by:

(1) for the surface stratum:

$$U_1(z_1) = E_1 \cdot e^{i\kappa_1 z_1} + F_1 \cdot e^{-i\kappa_1 z_1} \quad (43)$$

$$\tau_1(z_1) = i(\rho_1 V_1)\omega(E_1 \cdot e^{i\kappa_1 z_1} - F_1 \cdot e^{-i\kappa_1 z_1}) \quad (44)$$

(2) for the engineering bedrock:

$$U_2(z_2) = E_o \cdot e^{i\kappa_2 z_2} + F_2 \cdot e^{-i\kappa_2 z_2} \quad (45)$$

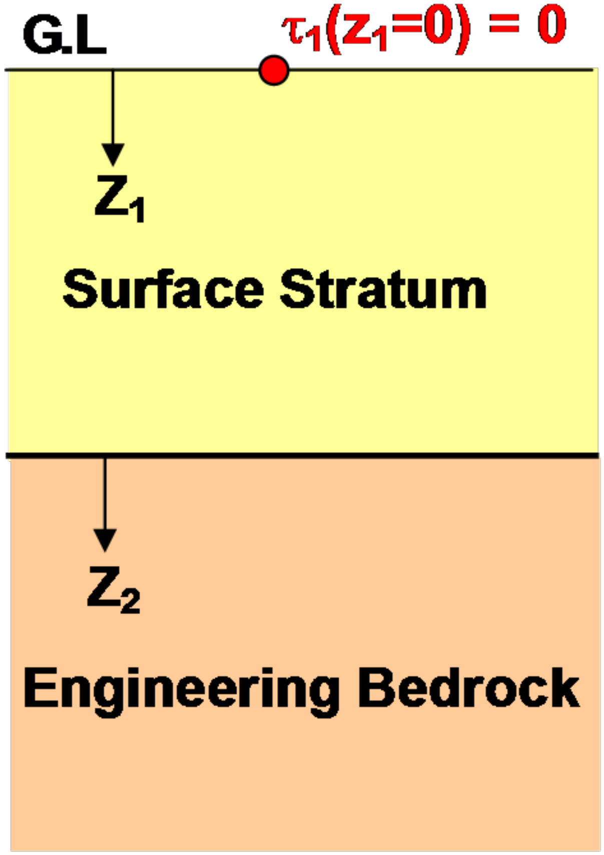
$$\tau_2(z_2) = i(\rho_2 V_2)\omega(E_o \cdot e^{i\kappa_2 z_2} - F_2 \cdot e^{-i\kappa_2 z_2}) \quad (46)$$

The subscripts 1 and 2 indicate the surface stratum and the engineering bedrock, respectively.

The shear stress at the ground surface ($z_1=0$) becomes zero.

$$\tau_1(z_1=0)=i(\rho_1 V_1)\omega(E_1-F_1)=0 \quad (47)$$

$$\therefore \quad E_1=F_1 \quad (48)$$

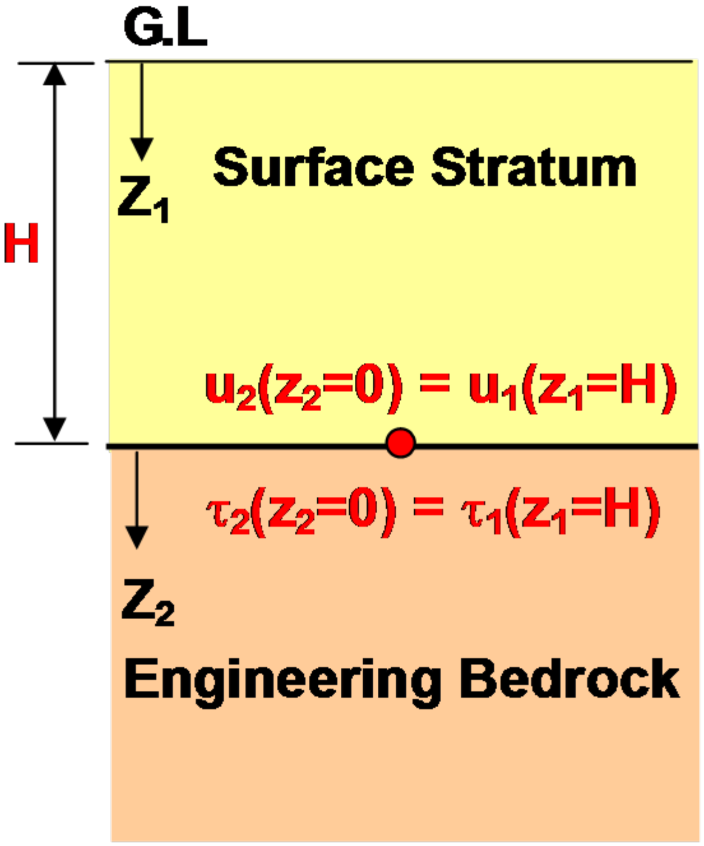


Therefore, the displacement and the shear stress in the surface stratum are:

$$U_1(z_1)=E_1(e^{i\kappa_1 z_1}+e^{-i\kappa_1 z_1}) \quad (49)$$

$$\tau_1(z_1)=i(\rho_1 V_1)\omega E_1(e^{i\kappa_1 z_1}-e^{-i\kappa_1 z_1}) \quad (50)$$

The boundary conditions are given at the interface between the surface stratum and the engineering bedrock.



$$U_2(z_2=0)=U_1(z_1=H) \qquad (51)$$

$$\tau_2(z_2=0)=\tau_1(z_1=H) \qquad (52)$$

Substituting Eq.(45) and Eq.(49) into Eq.(51):

$$E_o+F_2=E_1(e^{i\kappa_1 H}+e^{-i\kappa_1 H}) \qquad (53)$$

Putting Eq.(46) and Eq.(50) into Eq.(52):

$$i(\rho_2 V_2)\omega(E_o-F_2)=i(\rho_1 V_1)\omega E_1(e^{i\kappa_1 H}-e^{-i\kappa_1 H}) \qquad (54)$$

$$\therefore \quad E_o-F_2=\frac{\rho_1 V_1}{\rho_2 V_2}E_1(e^{i\kappa_1 H}-e^{-i\kappa_1 H})=\alpha E_1(e^{i\kappa_1 H}-e^{-i\kappa_1 H}) \quad (55)$$

where,

$$\alpha=\frac{\rho_1 V_1}{\rho_2 V_2} \qquad (56)$$

Eq.(53) : $$E_o + F_2 = E_1(e^{i\kappa_1 H} + e^{-i\kappa_1 H}) \quad (53)$$

plus Eq.(55) : $$E_o - F_2 = \alpha E_1(e^{i\kappa_1 H} - e^{-i\kappa_1 H}) \quad (55)$$

gives,

$$2E_o = E_1\{(1+\alpha)e^{i\kappa_1 H} + (1-\alpha)e^{-i\kappa_1 H}\} \quad (57)$$

$$\therefore \quad E_1 = \frac{2E_o}{(1+\alpha)e^{i\kappa_1 H} + (1-\alpha)e^{-i\kappa_1 H}} \quad (58)$$

From Eq.(53): $$E_o + F_2 = E_1(e^{i\kappa_1 H} + e^{-i\kappa_1 H})$$

$$F_2 = E_1(e^{i\kappa_1 H} + e^{-i\kappa_1 H}) - E_o$$

Putting Eq.(58) into the above equation,

$$F_2 = \frac{2E_o(e^{i\kappa_1 H} + e^{-i\kappa_1 H})}{(1+\alpha)e^{i\kappa_1 H} + (1-\alpha)e^{-i\kappa_1 H}} - E_o$$

$$= \frac{E_o[2(e^{i\kappa_1 H} + e^{-i\kappa_1 H}) - (1+\alpha)e^{i\kappa_1 H} - (1-\alpha)e^{-i\kappa_1 H}]}{(1+\alpha)e^{i\kappa_1 H} + (1-\alpha)e^{-i\kappa_1 H}}$$

$$= \frac{E_o[(1-\alpha)e^{i\kappa_1 H} + (1+\alpha)e^{-i\kappa_1 H}]}{(1+\alpha)e^{i\kappa_1 H} + (1-\alpha)e^{-i\kappa_1 H}} \qquad (59)$$

The displacement U_s at the ground surface:

$$U_S = U_1(z_1 = 0) = 2E_1$$

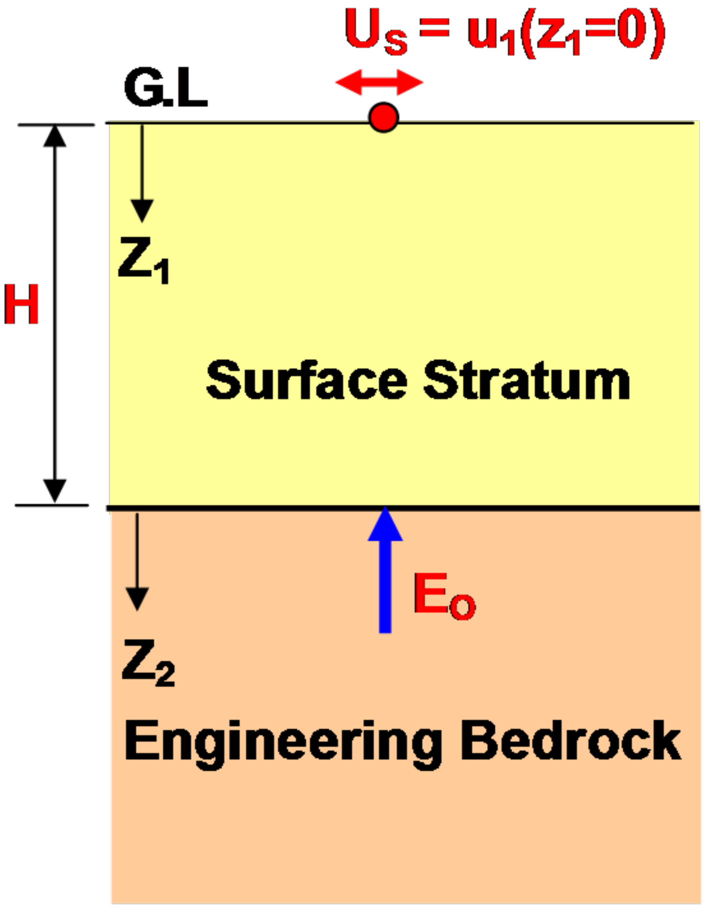
$$= \frac{4E_o}{(1+\alpha)e^{i\kappa_1 H} + (1-\alpha)e^{-i\kappa_1 H}}$$

$$= \frac{4E_o}{(e^{i\kappa_1 H} + e^{-i\kappa_1 H}) + \alpha(e^{i\kappa_1 H} - e^{-i\kappa_1 H})}$$

$$= \frac{4E_o}{2\cos(\kappa_1 H) + i2\alpha\sin(\kappa_1 H)}$$

$$= \frac{2E_o}{\cos(\kappa_1 H) + i\alpha\sin(\kappa_1 H)} \qquad (60)$$

$$\therefore \quad \frac{U_S}{2E_o} = \frac{1}{\cos(\kappa_1 H) + i\alpha\sin(\kappa_1 H)} \qquad (61)$$

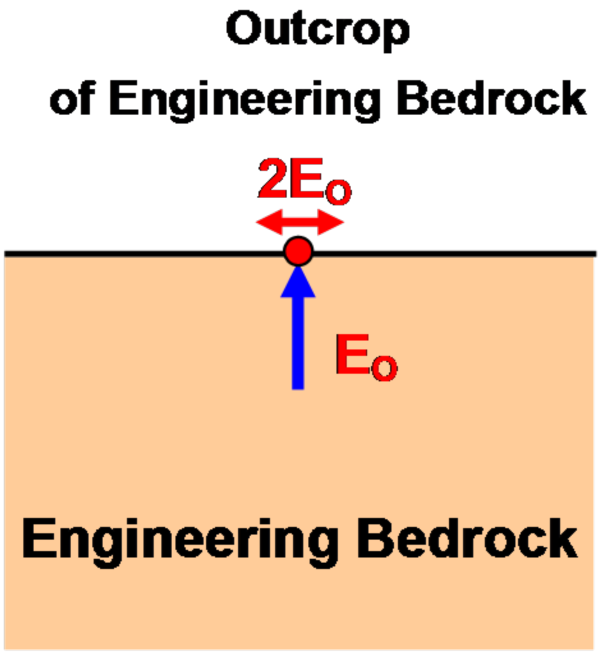


Absolute value of $U_S/(2E_O)$ is:

$$\text{abs.}\left(\frac{U_S}{2E_o}\right) = \frac{1}{\sqrt{\cos^2(\kappa_1 H) + \alpha^2 \sin^2(\kappa_1 H)}}$$

$$= \frac{1}{\sqrt{\cos^2\left(\frac{\omega H}{V_1}\right) + \alpha^2 \sin^2\left(\frac{\omega H}{V_1}\right)}} \qquad (62)$$

$2E_O$ is the displacement of the engineering bedrock, when the surface stratum is removed and the engineering bedrock is in outcrop.

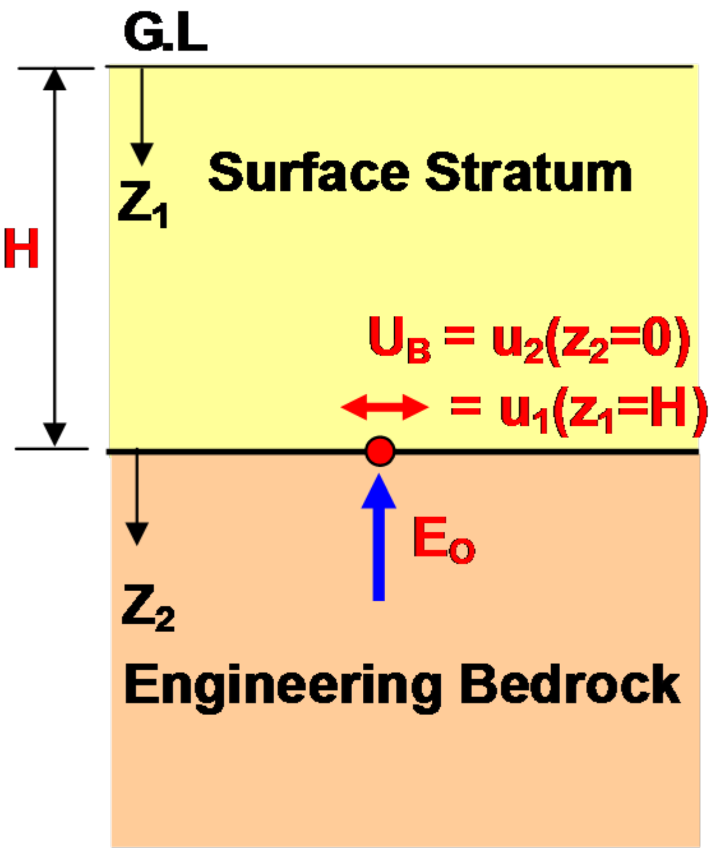


Similarly, the displacement at the interface is given by:

$$U_B = U_2(z_2 = 0) = E_o + F_2$$

$$= \frac{2E_o \cos(\kappa_1 H)}{\cos(\kappa_1 H) + i\alpha \sin(\kappa_1 H)}$$

$$\therefore \quad U_B = \frac{2E_o}{1 + i\alpha \tan(\kappa_1 H)} \qquad (63)$$

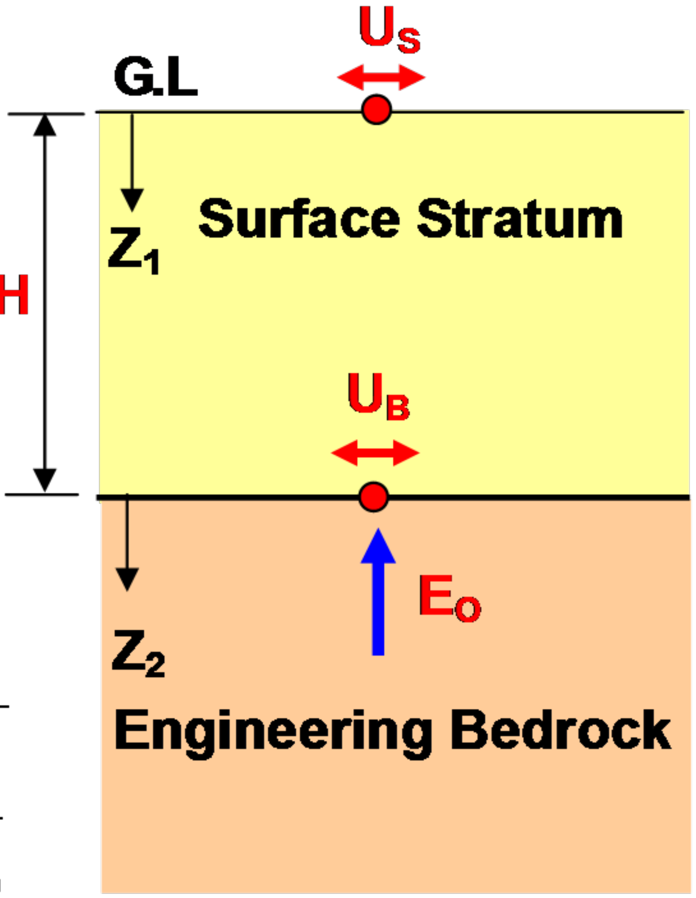


Absolute value of $U_B/(2E_O)$ is:

$$\text{Abs.}\left(\frac{U_B}{2E_o}\right) = \frac{1}{\sqrt{1 + \alpha^2 \tan^2(\kappa_1 H)}} = \frac{1}{\sqrt{1 + \alpha^2 \tan^2\left(\frac{\omega H}{V_1}\right)}} \qquad (64)$$

The ratio of U_S to U_B is :

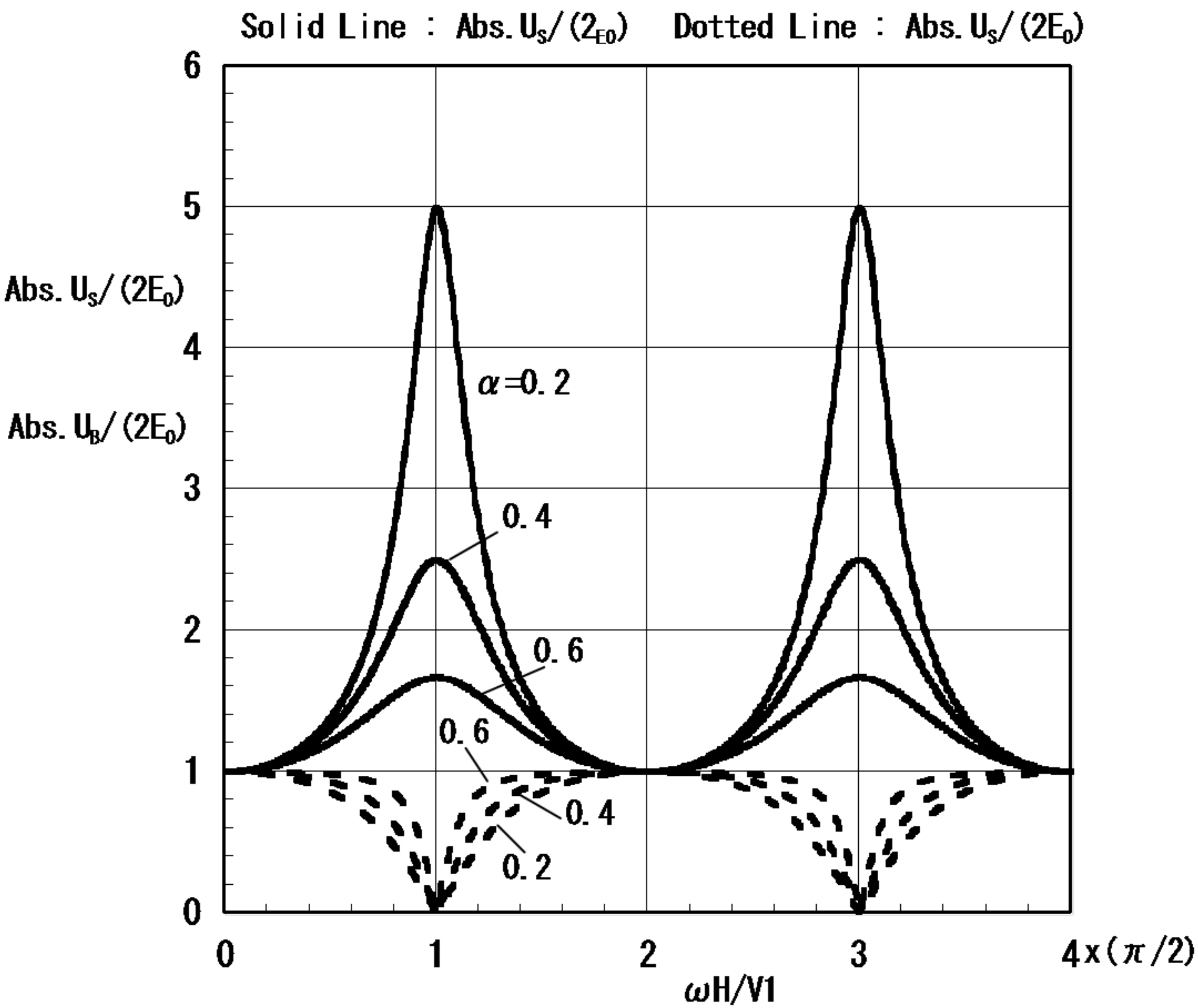
$$\frac{U_S}{U_B} = \frac{1 + i\alpha \tan(\kappa_1 H)}{\cos(\kappa_1 H) + i\alpha \sin(\kappa_1 H)} \qquad (65)$$



And the absolute value is:

$$Abs.\left(\frac{U_S}{U_B}\right) = \sqrt{\frac{1 + \alpha^2 \tan^2(\kappa_1 H)}{\cos^2(\kappa_1 H) + \alpha^2 \sin^2(\kappa_1 H)}}$$

$$= \sqrt{\frac{1 + \alpha^2 \tan^2\left(\frac{\omega H}{V_1}\right)}{\cos^2\left(\frac{\omega H}{V_1}\right) + \alpha^2 \sin^2\left(\frac{\omega H}{V_1}\right)}} \qquad (66)$$



Abs.($U_S/2E_O$) and Abs.($U_B/2E_O$)

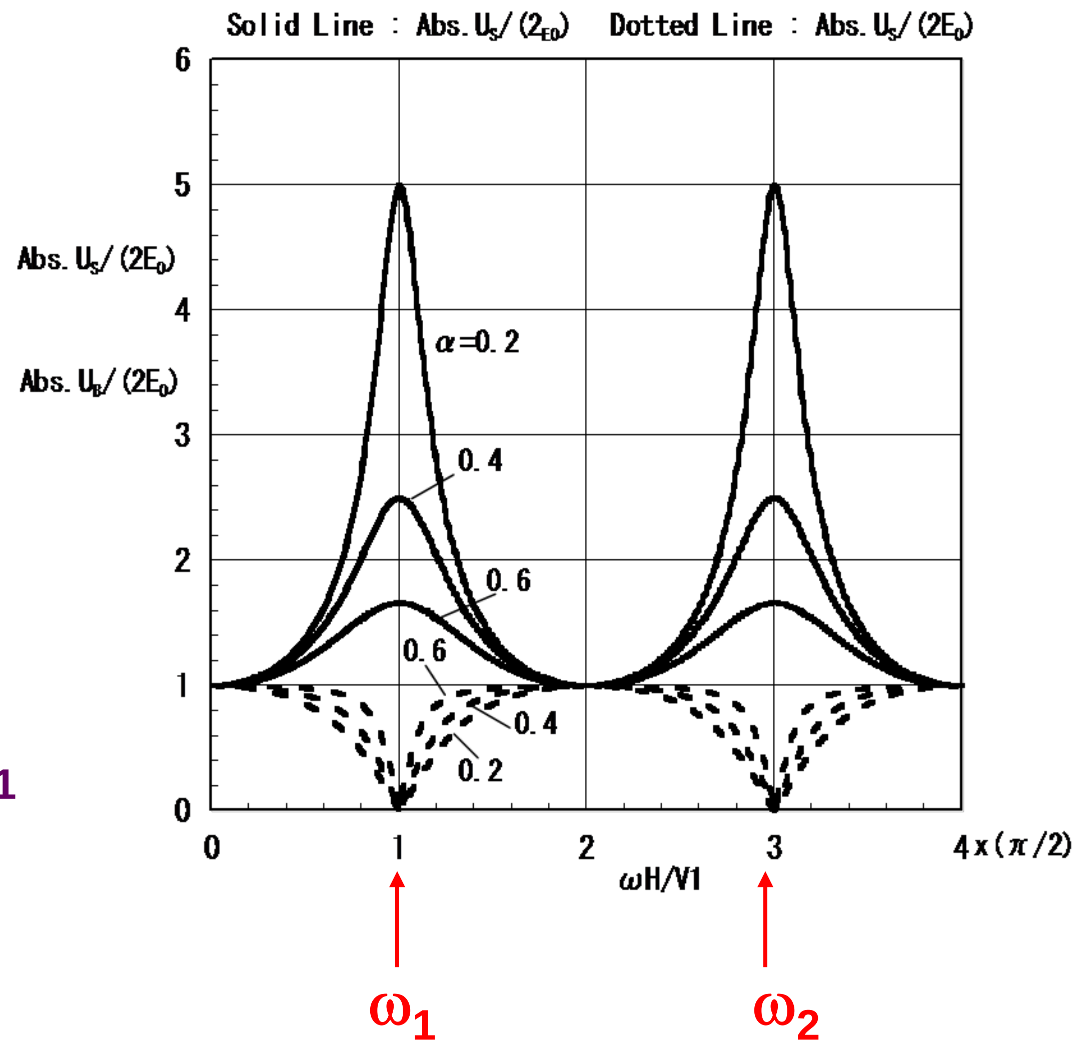
The first natural circular frequency:

$$\omega_1 H/V_1=\pi/2$$

4 $\omega_1=(\pi/2)(V_1/H)$

The second natural circular frequency:

$$\omega_2=3\,\omega_1$$



The first natural period T_1 and frequency f_1:

$$T_1=2\pi/\,\omega_1=(4H)/\,V_1$$

$$f_1=1/\,T_1=V_1/(4H)$$

The second natural period T_2 and frequency f_2:

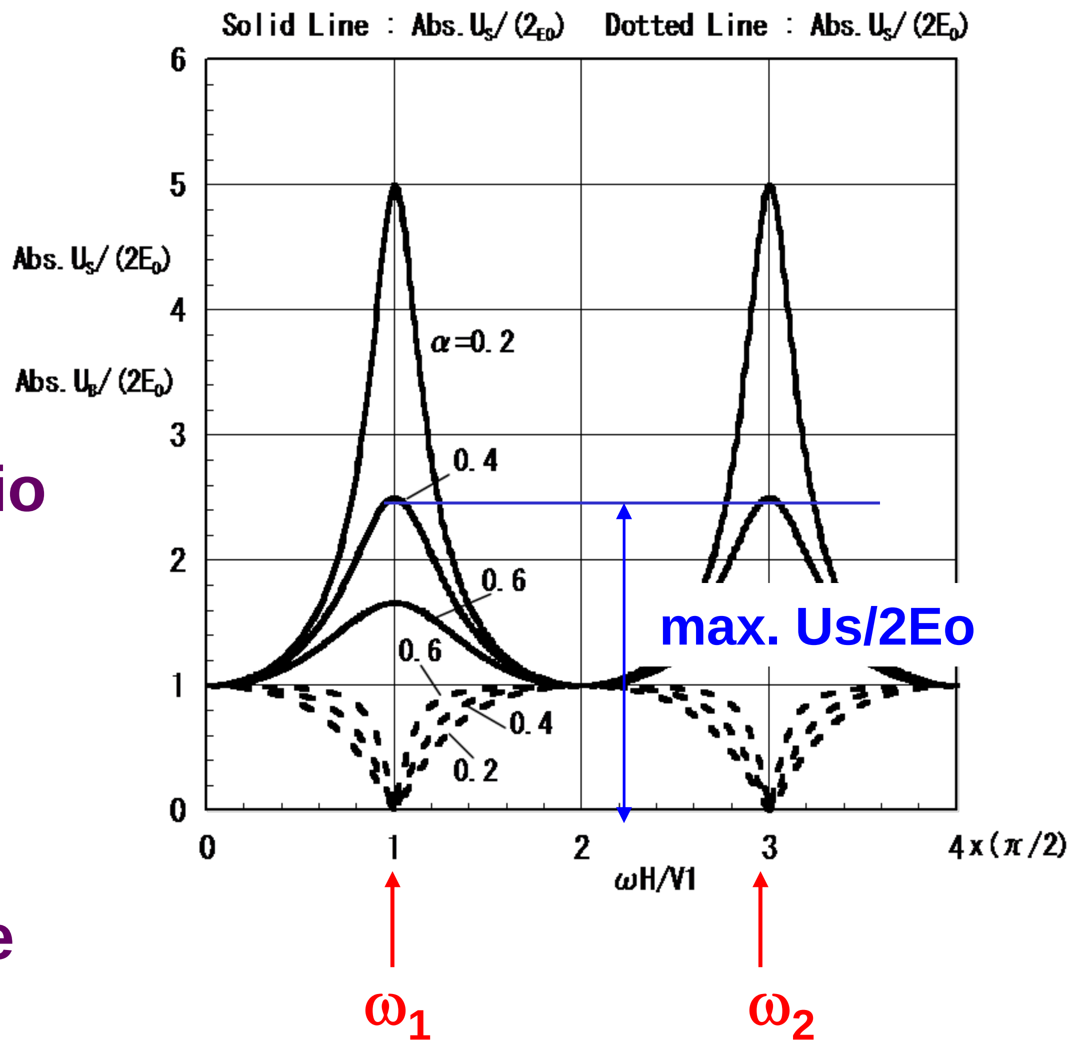
$$T_2=T_1/3 \qquad f_2=3f_1$$

Abs. $U_S/2E_O$ at the natural frequency is:

$1/\alpha$

When the impedance ratio α is smaller, the amplification at the natural frequency becomes larger.

That is, when the surface stratum is much softer than the engineering bedrock, the amplification becomes much larger.



E N D

One-Dimensional Shear Wave Propagation

in Two-Layered Strata

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