Dynamic Soil Structure Interaction

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One-Dimensional Shear Wave Propagation In Multi-Layered Strata (SHAKE)

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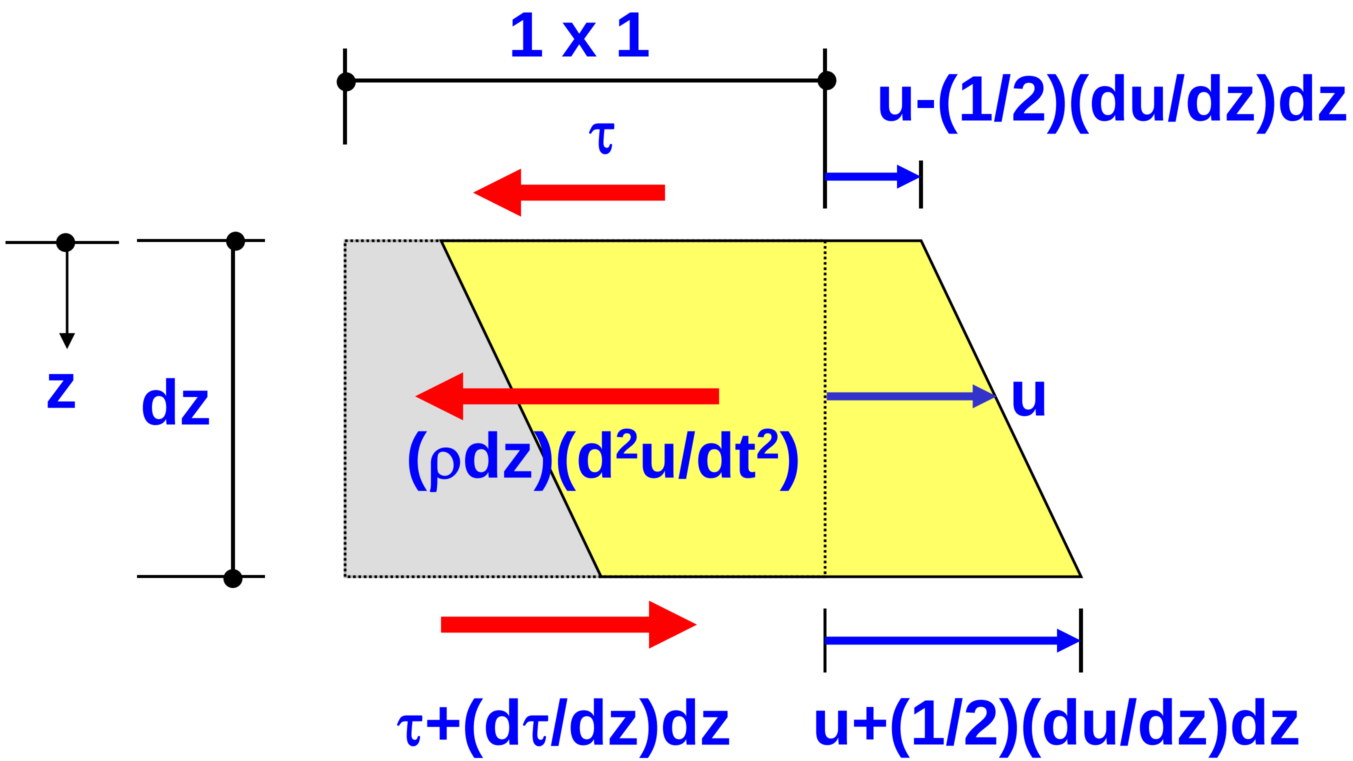
SH-Shear Wave Propagation in Multi-Layered Strata

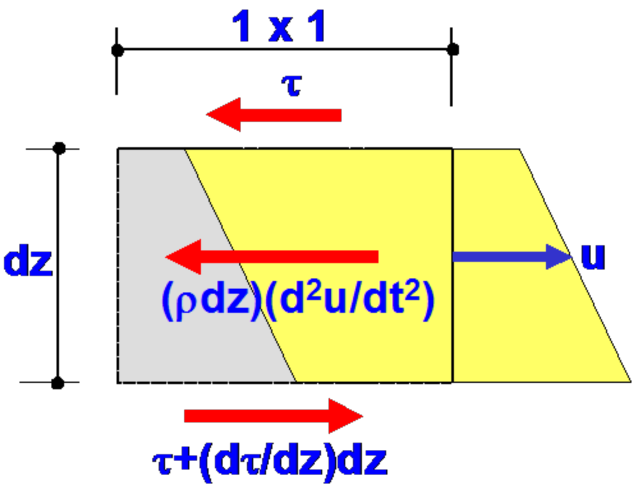
(1) Equation of Shear Wave Propagation



Equilibrium of Forces

Equilibrium of forces acting on the small cube in a shear column with unit area:

$$\left(\tau + \frac{\partial \tau}{\partial z} dz\right) - \tau - (\rho dz)\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$



$$\therefore \qquad \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau}{\partial z} \qquad (2)$$

where u, τ and ρ denote the displacement, shear stress and mass density, respectively.

The shear stress τ is expressed by:

$$\tau = G^* \frac{\partial u}{\partial z} \qquad (3)$$

Substituting Eq.(3) into Eq.(2), then

$$\rho \frac{\partial^2 u}{\partial t^2} = G^* \frac{\partial^2 u}{\partial z^2} \qquad (4)$$

Eq.(4) is the equation of motion for the shear wave propagation.

Where,

$$G^* = G(1 + i2h) \qquad (5)$$

$$G^* / \rho = G(1 + i2h) / \rho = V^2 (1 + i2h) \qquad (6)$$

G^* : **Complex Elastic Shear Modulus**

$$G^* = G + iG' = G(1 + i\frac{G'}{G}) = G(1 + i2h)$$

G : **Elastic shear Modulus**

ρ : **Mass Density**

h : **Damping Factor(** $= G' / (2G)$ **)**

$V = \sqrt{G / \rho}$: **Shear Wave Propagation Velocity**

$i = \sqrt{-1}$: **Complex Unit**

(2) Solution of the Equation of Wave Propagation

We assume the displacement u is harmonic motion with the amplitude U and the angular frequency ω .

That is,

$$u = U(z)e^{i\omega t} = U(z)\cdot(\cos\omega t + i\sin\omega t) \qquad (7)$$

Substituting Eq.(7) into Eq.(4) :

$$\rho\frac{\partial^2 u}{\partial t^2} = G^*\frac{\partial^2 u}{\partial z^2} \qquad (4)$$

we obtain the equation below:

$$\frac{d^2U}{dz^2} + \frac{\omega^2}{V^2(1+i2h)}U = 0 \qquad (8)$$

Introducing the notation:

$$\varsigma^2 = 1 + i2h \qquad (9)$$

Then, the complex elastic shear modulus can be expressed by:

$$G^* = G(1 + i2h) = G\zeta^2 = \rho V^2 \zeta^2 \qquad (10)$$

ζ is complex number. We adopt ζ of which both real and imaginary part are positive real numbers. That is,

$$Real(\varsigma) \geq 0 \qquad Imag.(\varsigma) \geq 0 \qquad (11)$$

In the left hand side of Eq.(8):

$$\frac{d^2U}{dz^2} + \frac{\omega^2}{V^2(1 + i2h)} U = 0 \qquad (8)$$

Expressing the coefficient of the second term by:

$$\frac{\omega^2}{V^2}\frac{1}{1+i2h} = \left(\frac{\omega}{V\varsigma}\right)^2 = \kappa^2 \qquad (12)$$

Then, Eq.(8):

$$\frac{d^2U}{dz^2} + \frac{\omega^2}{V^2(1+i2h)}U = 0 \qquad (8)$$

is transformed into Eq.(13):

$$\frac{d^2U}{dz^2} + \kappa^2 U = 0 \qquad (13)$$

The solution of Eq.(13) is given by:

$$U = E e^{i\kappa z} + F e^{-i\kappa z} \qquad (14)$$

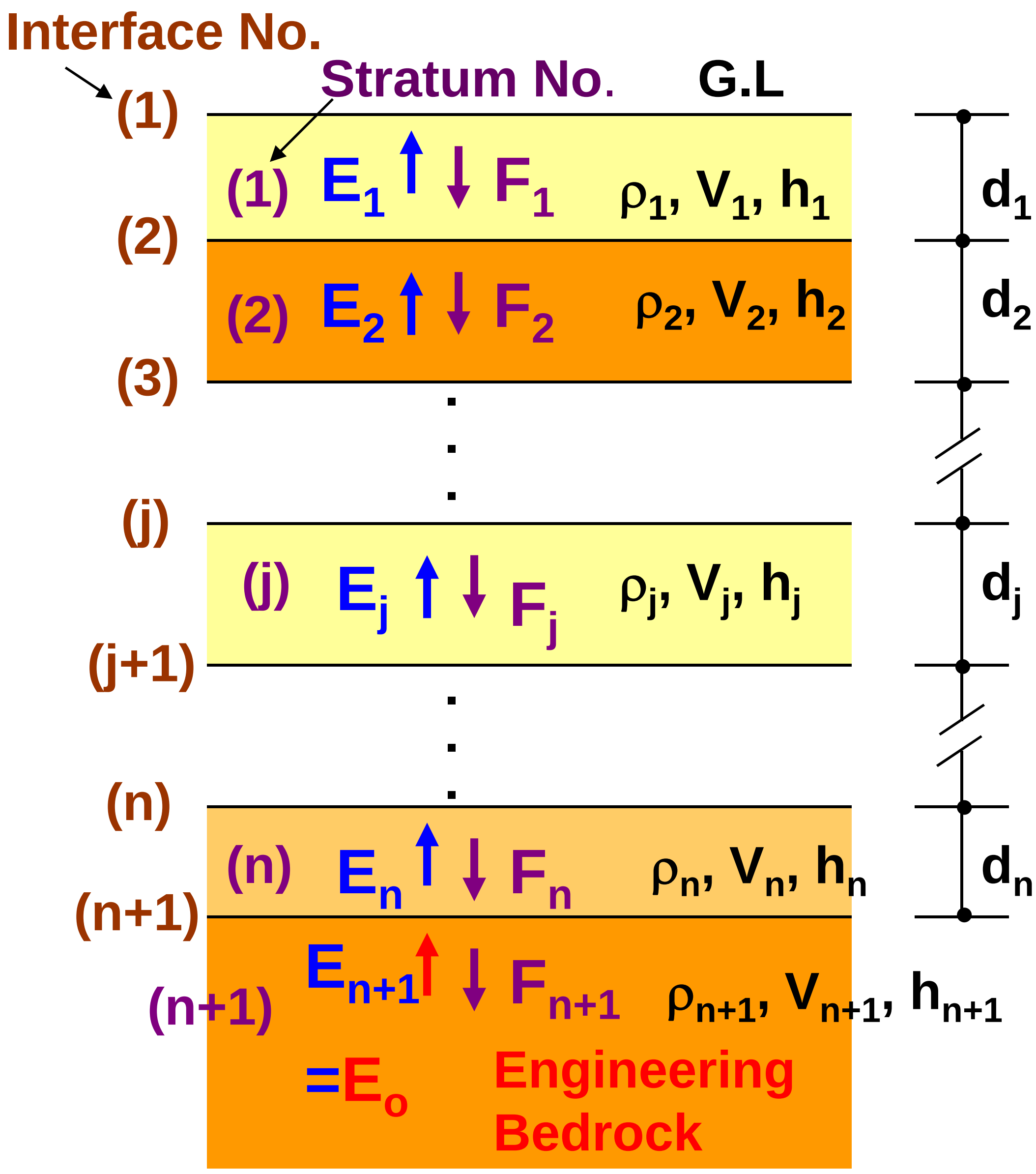
The shear stress(amplitude) T, which corresponds to the displacement(amplitude) U, is expressed by :

$$T = G^* \frac{dU}{dz} = iG^* \kappa \{E e^{i\kappa z} - F e^{-i\kappa z}\} \qquad (15)$$

(3) Application to Multi-Layered Strata

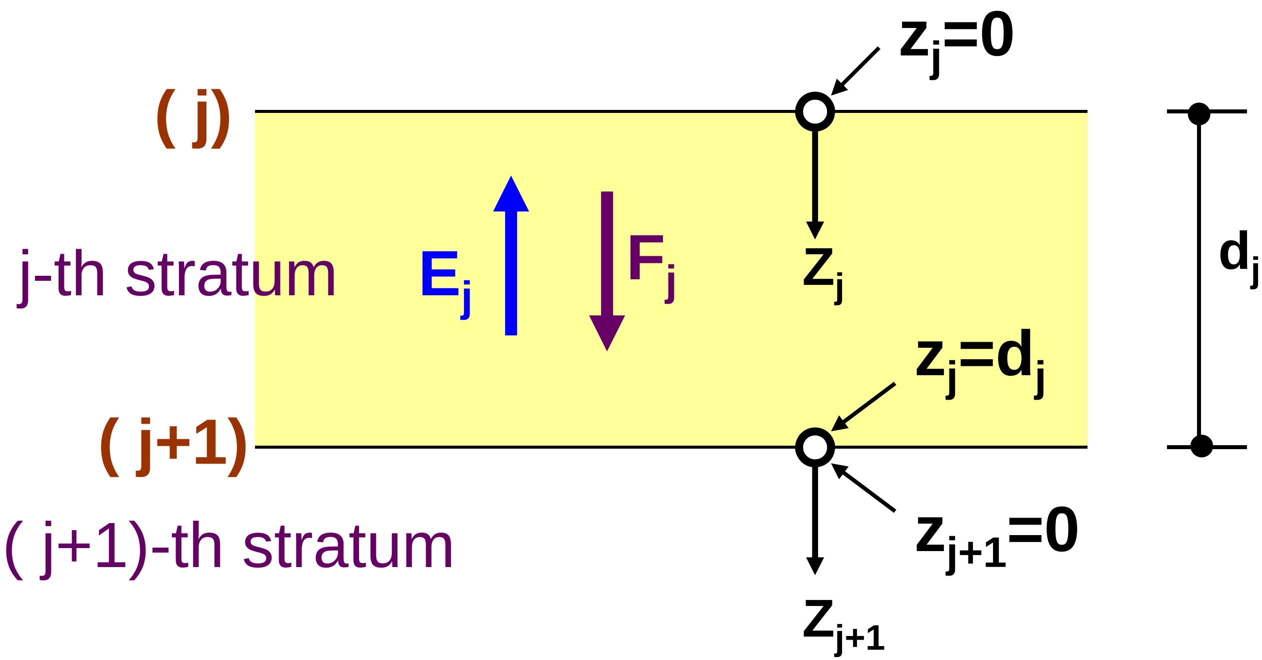
On the multi-layered strata shown in the figure, we consider the wave propagation due to the incident earthquake wave $E_O e^{i\omega t}$, which propagates upward in the deepest stratum (Engineering Bedrock).

$E_O e^{i\omega t}$ indicates the displacement of the incident wave with the amplitude E_O and the angular frequency ω.



Wave Propagation in Multi-Layered Strata

We consider the wave propagation in j-th stratum of the multi-layered strata.



The displacement amplitude U and the shear stress amplitude T are expressed by Eq.(14) and Eq.(15) as explained before.

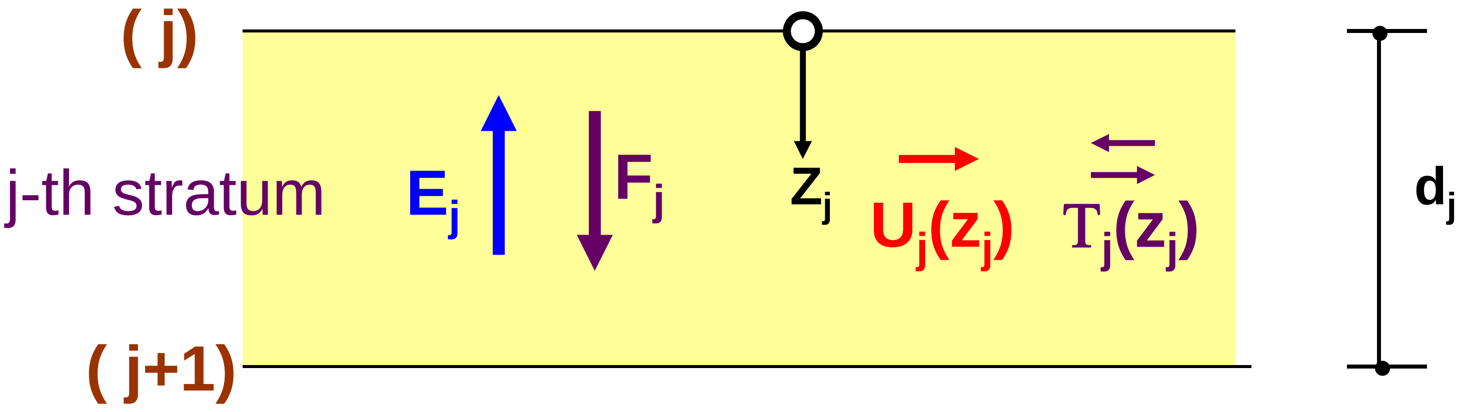
$$U = E\,e^{i\kappa z} + F\,e^{-i\kappa z} \tag{14}$$

$$T = G^{*}\frac{dU}{dz} = iG^{*}\kappa\{E\,e^{i\kappa z} - F\,e^{-i\kappa z}\} \tag{15}$$

The subscript j is employed for expression of the term concerning j-th stratum. The displacement amplitude $U_j(z_j)$ and the shear stress amplitude $T_j(z_j)$ in the j-th stratum can be expressed by:

$$U_j(z_j) = E_j e^{i\kappa_j z_j} + F_j e^{-i\kappa_j z_j} \quad (16)$$

$$T_j(z_j) = iG_j^* \kappa_j \{E_j e^{i\kappa_j z_j} - F_j e^{-i\kappa_j z_j}\} \quad (17)$$



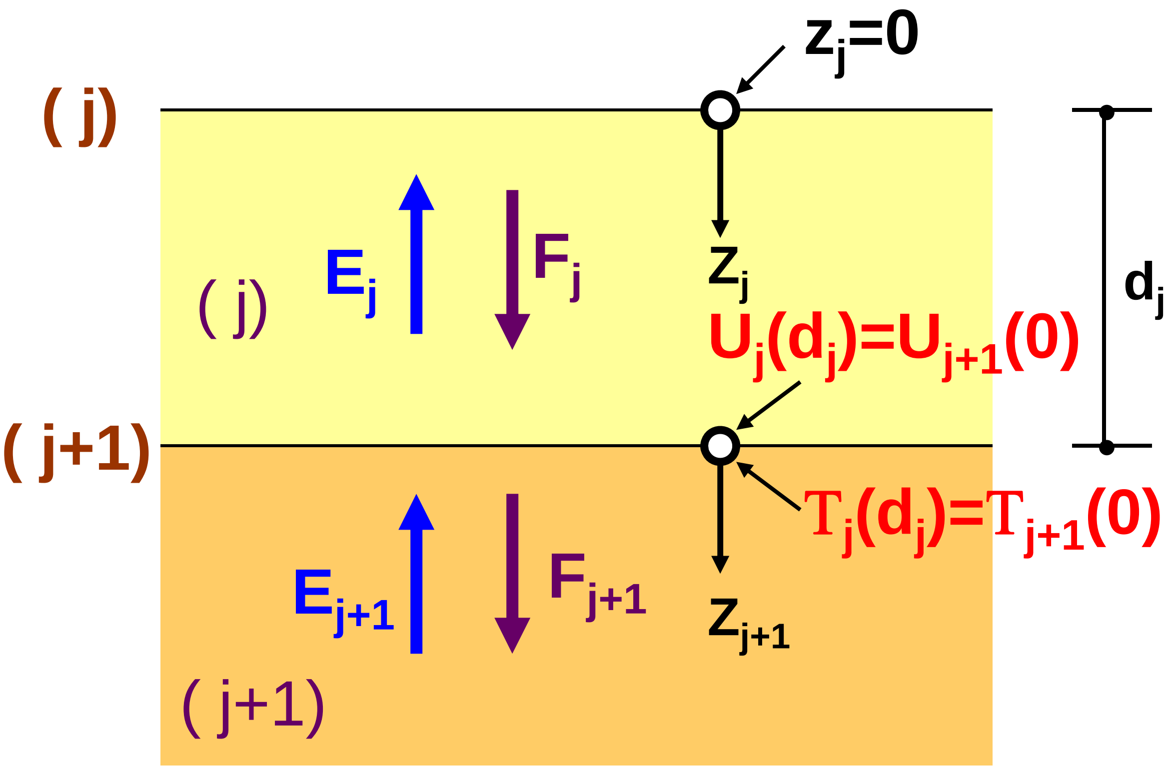
On the (j+1)-th interface between j-th and (j+1)-th stratum, the displacement continuity:

$$U_j(d_j) = U_{j+1}(0) \tag{18}$$

And the shear stress continuity :

$$T_j(d_j) = T_{j+1}(0) \tag{19}$$

must be satisfied:



Putting Eq.(16) :

$$U_j(z_j) = E_j e^{i\kappa_j z_j} + F_j e^{-i\kappa_j z_j} \qquad (16)$$

into Eq.(18) :

$$U_j(d_j) = U_{j+1}(0) \qquad (18)$$

gives:

$$E_{j+1} + F_{j+1} = E_j e^{ik_j d_j} + F_j e^{-ik_j d_j} \qquad (20)$$

And also, putting Eq.(17) :

$$T_j(z_j) = iG_j^* \kappa_j \{E_j e^{i\kappa_j z_j} - F_j e^{-i\kappa_j z_j}\} \qquad (17)$$

into Eq.(19) :

$$T_j(d_j) = T_{j+1}(0) \qquad (19)$$

gives:

$$E_{j+1} - F_{j+1} = \frac{G_j^* k_j}{G_{j+1}^* k_{j+1}} \{E_j e^{ik_j d_j} - F_j e^{-ik_j d_j}\} \qquad (21)$$

The coefficient in the right hand side of Eq.(21) can be expressed by:

$$\alpha_j = \frac{G_j^* k_j}{G_{j+1}^* k_{j+1}} = \frac{\rho_j V_j}{\rho_{j+1} V_{j+1}} \frac{\varsigma_j}{\zeta_{j+1}} \qquad (22)$$

α_j denotes the ratio of complex wave impedance between j-th and (j+1)-th stratum.

Using α_j , Eq.(21) is transformed into Eq.(23):

$$E_{j+1} - F_{j+1} = \alpha_j \{E_j e^{ik_j d_j} - F_j e^{-ik_j d_j}\} \qquad (23)$$

Eq.(23) plus Eq.(20):

$$E_{j+1} + F_{j+1} = E_j e^{ik_j d_j} + F_j e^{-ik_j d_j} \qquad (20)$$

gives:

$$2E_{j+1} = (1+\alpha_j)e^{ik_j d_j} E_j + (1-\alpha_j)e^{-ik_j d_j} F_j \qquad (24)$$

Eq.(20) minus Eq.(23) gives:

$$2F_{j+1} = (1-\alpha_j)e^{ik_j d_j} E_j + (1+\alpha_j)e^{-ik_j d_j} F_j \qquad (25)$$

Eq.(24) and (25):

$$2E_{j+1} = (1+\alpha_j)e^{ik_jd_j}E_j + (1-\alpha_j)e^{-ik_jd_j}F_j \qquad (24)$$

$$2F_{j+1} = (1-\alpha_j)e^{ik_jd_j}E_j + (1+\alpha_j)e^{-ik_jd_j}F_j \qquad (25)$$

are rearranged into the matrix-vector formulation as:

$$\begin{Bmatrix} E_{j+1} \\ F_{j+1} \end{Bmatrix} = [A_j] \begin{Bmatrix} E_j \\ F_j \end{Bmatrix} \qquad (26)$$

where,

$$[A_j] = \frac{1}{2} \begin{bmatrix} (1+\alpha_j)e^{ik_jd_j} & (1-\alpha_j)e^{-ik_jd_j} \\ (1-\alpha_j)e^{ik_jd_j} & (1+\alpha_j)e^{-ik_jd_j} \end{bmatrix} \qquad (27)$$

For simplicity, putting:

$$\{C_j\} = \begin{Bmatrix} E_j \\ F_j \end{Bmatrix} \qquad (28)$$

then, Eq.(26) :

$$\begin{Bmatrix} E_{j+1} \\ F_{j+1} \end{Bmatrix} = [A_j] \begin{Bmatrix} E_j \\ F_j \end{Bmatrix} \qquad (26)$$

can be expressed by:

$$\{C_{j+1}\} = [A_j]\{C_j\} \qquad (29)$$

Eq.(29): $\{C_{j+1}\} = [A_j]\{C_j\}$ (29)

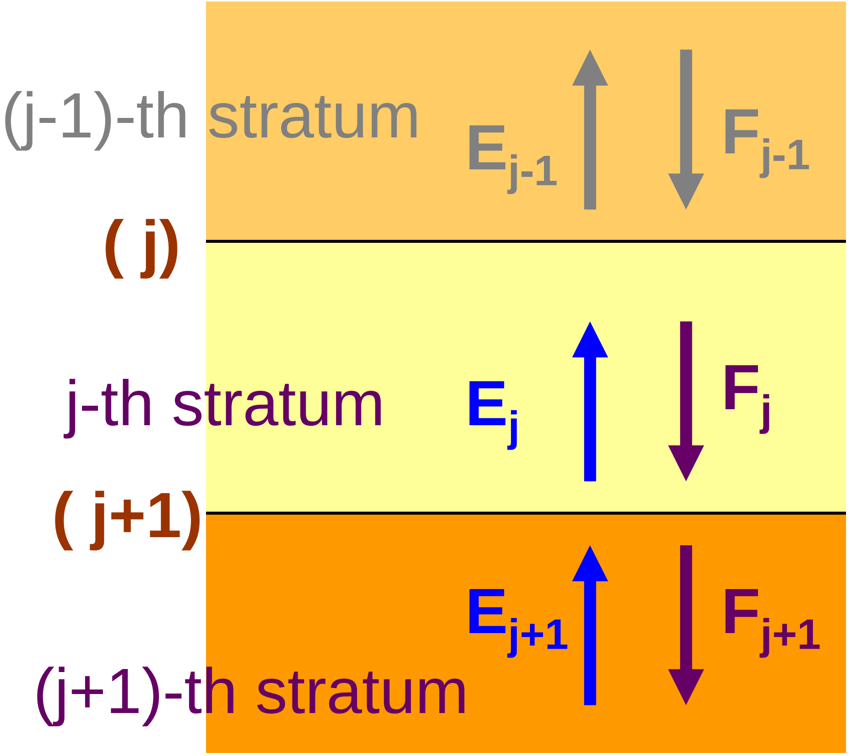
represents the relation of the amplitude vector of the waves propagating upward and downward

between in (j+1)-th stratum:

$$\{C_{j+1}\} = \{E_{j+1}\ F_{j+1}\}^T \qquad (30)$$

and these in j-th stratum:

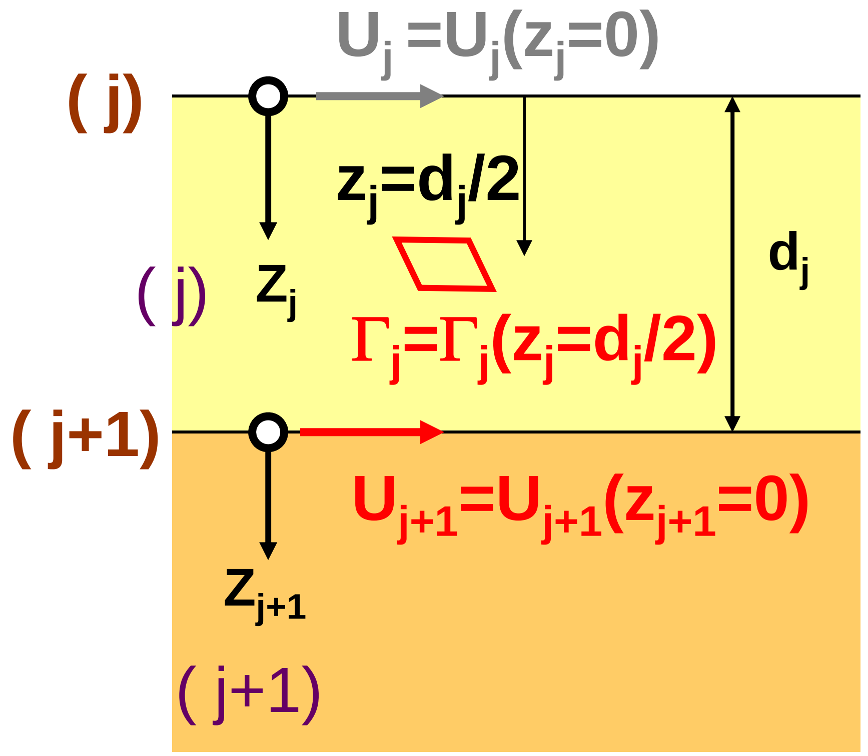
$$\{C_j\} = \{E_j\ F_j\}^T \qquad (31)$$



The displacement amplitude at the (j+1)-th interface between j-th stratum and (j+1)-th stratum is given by:

$$U_{j+1}=U_{j+1}(z_{j+1}=0)=E_{j+1}+F_{j+1} \qquad (32)$$

The shear strain amplitude Γ_j in the j-th stratum is evaluated at the middle depth ($z_j=d_j/2$) of the stratum.



That is,

$$\Gamma_j = [\frac{dU_j(z_j)}{dz_j}]_{z_j=d_j/2}$$

$$= ik_j\{E_j e^{i(k_j d_j/2)} - F_j e^{-i(k_j d_j/2)}\} \qquad (33)$$

In Eq.(29):

$$\{C_{j+1}\} = [A_j]\{C_j\} \qquad (29)$$

putting j=n, then:

$$\{C_{n+1}\} = [A_n]\{C_n\} \qquad (34)$$

Furthermore, putting j=n-1,

$$\{C_n\} = [A_{n-1}]\{C_{n-1}\} \qquad (35)$$

Substituting Eq.(35) into Eq.(34), then:

$$\{C_{n+1}\} = [A_n][A_{n-1}]\{C_{n-1}\} \qquad (36)$$

Repeating the calculations from Eq.(34) to Eq.(36) up to j=1,

$$\{C_{n+1}\} = [A_n]\{C_n\} \qquad (34)$$

$$\{C_n\} = [A_{n-1}]\{C_{n-1}\} \qquad (35)$$

$$\{C_{n+1}\} = [A_n][A_{n-1}]\{C_{n-1}\} \qquad (36)$$

we obtain:

$$\{C_{n+1}\} = [A_n][A_{n-1}]\{C_{n-1}\} = [A_n][A_{n-1}][A_{n-2}]\{C_{n-2}\}$$
$$= \quad \cdot \quad \cdot \quad \cdot \quad = [A_n][A_{n-1}][A_{n-2}]\cdots[A_2][A_1]\{C_1\} \qquad (37)$$

Putting,

$$[B] = [A_n][A_{n-1}][A_{n-2}]\cdots[A_2][A_1] \qquad (38)$$

then, Eq.(37) can be rewritten by:

$$\{C_{n+1}\} = [B]\{C_1\} \qquad (39)$$

Expressing Eq.(39) :

$$\{C_{n+1}\} = [B]\{C_1\} \tag{39}$$

by the elements in the vectors and the matrix, then

$$\begin{Bmatrix} E_{n+1} = E_o \\ F_{n+1} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} E_1 \\ F_1 \end{Bmatrix} \tag{40}$$

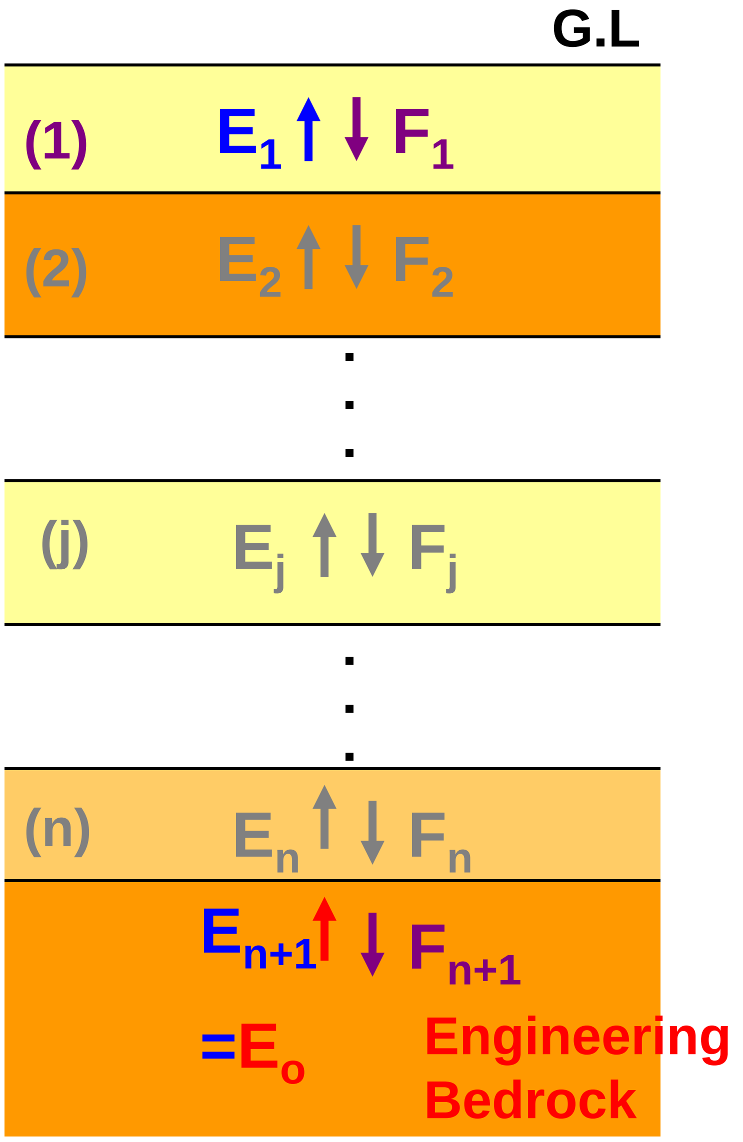
where,

$$[B] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{41}$$

E_O: the amplitude of incident wave

Eq.(39) or Eq.(40) :

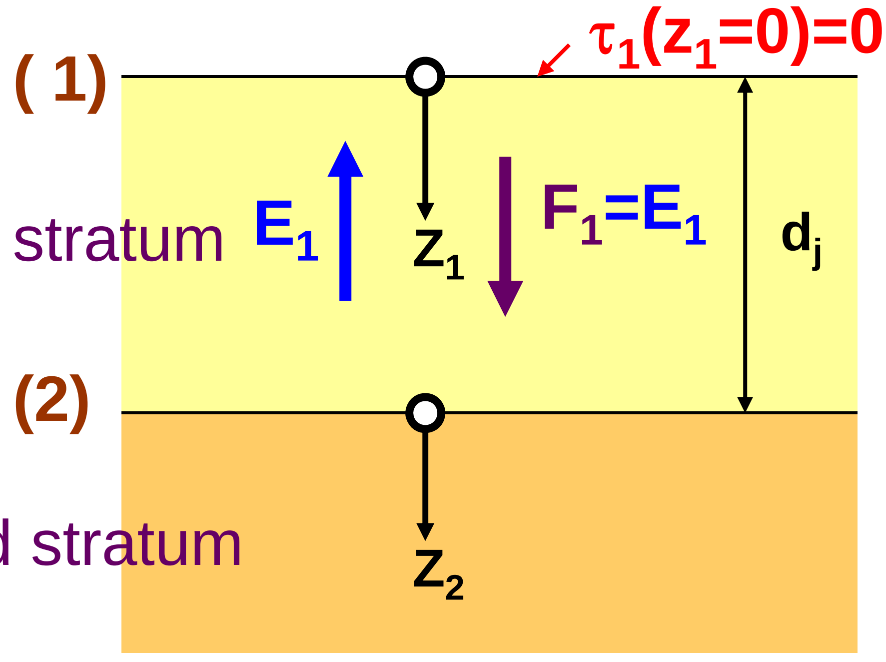
$$\begin{Bmatrix} E_{n+1} = E_o \\ F_{n+1} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} E_1 \\ F_1 \end{Bmatrix} \qquad (40)$$



express the relation of the displacement amplitudes between $\{E_{n+1}(=E_O)\ F_{n+1}\}^T$ of the waves in the deepest stratum (n+1) and $\{E_1\ F_1\}^T$ in the shallowest stratum 1.

In Eq.(40), E_O denotes the incident wave amplitude at the engineering bedrock.

Because the shear stress at the soil surface ($z_1=0$) must be null,



the shear stress amplitude given by Eq.(17):

$$T_j(z_j) = iG_j^* \kappa_j \{E_j e^{i\kappa_j z_j} - F_j e^{-i\kappa_j z_j}\} \qquad (17)$$

becomes zero at the soil surface $j=1$ and $z_1=0$.

$$T_1(z_1=0) = ik_1 G_1^* (E_1 - F_1) = 0 \qquad (42)$$

$$\therefore \quad E_1 = F_1 \qquad (43)$$

Substituting Eq.(43) into Eq.(16):

$$E_1 = F_1 \qquad (43)$$

$$U_j(z_j) = E_j e^{i\kappa_j z_j} + F_j e^{-i\kappa_j z_j} \qquad (16)$$

the displacement amplitude for the first stratum (j=1) becomes:

$$U_1(z_1) = E_1 e^{ik_1 z_1} + F_1 e^{-ik_1 z_1} = E_1 (e^{ik_1 z_1} + e^{-ik_1 z_1}) \qquad (44)$$

and the displacement amplitude Us at the soil surface z_1=0 becomes:

$$U_S = U_0(z_1 = 0) = 2E_1 \qquad (45)$$

Substituting Eq.(43):

$$E_1 = F_1 \qquad (43)$$

into Eq.(40) :

$$\begin{Bmatrix} E_{n+1} = E_o \\ F_{n+1} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} E_1 \\ F_1 \end{Bmatrix} \qquad (40)$$

then,

$$\begin{Bmatrix} E_{n+1} = E_o \\ F_{n+1} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_1 \end{Bmatrix} \qquad (46)$$

$$\therefore \qquad E_o = (b_{11} + b_{12})E_1 \qquad (47)$$

Eq.(47) gives:

$$E_1 = E_o / (b_{11} + b_{12}) \qquad (48)$$

Finally,

$$\{C_1\} = \begin{Bmatrix} E_1 \\ F_1 = E_1 \end{Bmatrix} = \begin{Bmatrix} 1/(b_{11} + b_{12}) \\ 1/(b_{11} + b_{12}) \end{Bmatrix} E_0$$

$$= \begin{Bmatrix} 0.5/(b_{11} + b_{12}) \\ 0.5/(b_{11} + b_{12}) \end{Bmatrix} (2E_0) \qquad (49)$$

Putting:

$$\{\tilde{C}_1\} = \begin{Bmatrix} \tilde{C}_1(1) \\ \tilde{C}_1(2) \end{Bmatrix} = \begin{Bmatrix} 0.5/(b_{11} + b_{12}) \\ 0.5/(b_{11} + b_{12}) \end{Bmatrix} \qquad (50)$$

then, Eq.(49) can be rewritten by:

$$\{C_1\} = \{\tilde{C}_1\}(2E_0) \qquad (51)$$

In Eq.(29) :

$$\{C_{j+1}\} = [A_j]\{C_j\} \qquad (29)$$

putting j=1, then:

$$\{C_2\} = [A_1]\{C_1\} = [A_1]\{\tilde{C}_1\}(2E_0) = \{\tilde{C}_2\}(2E_0) \qquad (52)$$

where,

$$\{\tilde{C}_2\} = [A_1]\{\tilde{C}_1\} \qquad (53)$$

Concerning the j-th stratum,

$$\{C_{j+1}\} = \{\tilde{C}_{j+1}\}(2E_0) \qquad (54)$$

$$\{\tilde{C}_{j+1}\} = [A_j]\{\tilde{C}_j\} \qquad (55)$$

the displacement amplitude U_j at the j-th interface is:

$$U_j = E_j + F_j = \{\tilde{C}_j(1) + \tilde{C}_j(2)\}(2E_o) = \tilde{U}_j \cdot (2E_o) \qquad (56)$$

The shear strain amplitude Γ_j in the j-th stratum is:

$$\Gamma_j = \Gamma_j(z_j = 0.5d_j)$$

$$= ik_j\{\tilde{C}_j(1)\exp(ik_j d_j/2) - \tilde{C}_j(2)\exp(-ik_j d_j/2)\}(2E_0) = \tilde{\Gamma}_j \cdot (2E_0) \quad (57)$$

where,

$$\tilde{U}_j = \tilde{C}_j(1) + \tilde{C}_j(2) \qquad (58)$$

$$\tilde{\Gamma}_j = ik_j\{\tilde{C}_j(1)\exp(ik_j d_j/2) - \tilde{C}_j(2)\exp(-ik_j d_j/2)\} \qquad (59)$$

(4) Time Domain Response Analysis

When the incident wave propagating upward in the engineering bedrock is assumed as the harmonic wave with the displacement amplitude of $E_o(\omega)$ and the angular frequency ω:

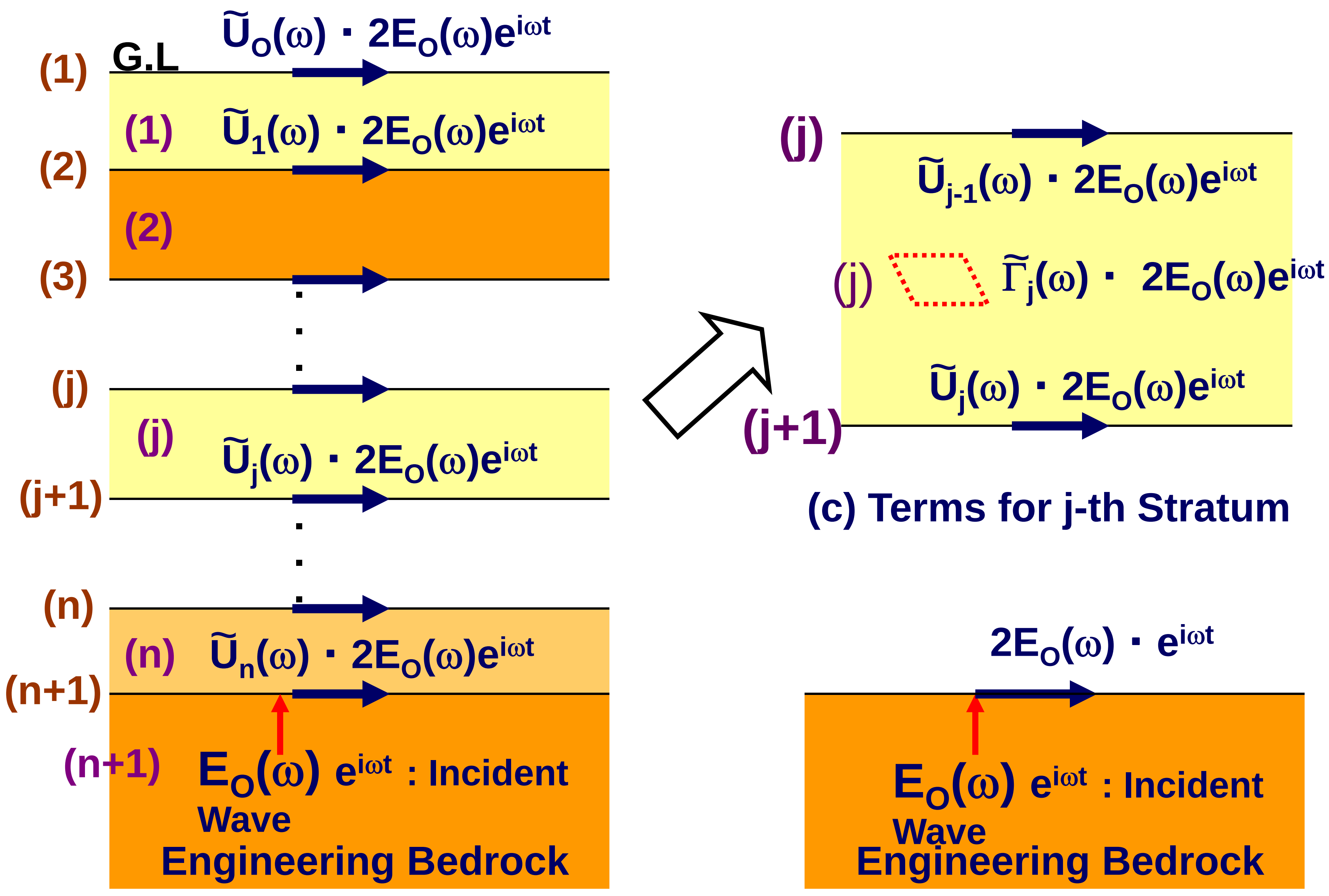
$$E_0(\omega)e^{i\omega t} \qquad (60)$$

the displacement amplitude at the j-th interface due to the incident wave is expressed by:

$$\tilde{U}_j(\omega)\cdot 2E_0(\omega)e^{i\omega t} \qquad (61)$$

and also, the shear strain amplitude by:

$$\tilde{\Gamma}_j(\omega)\cdot 2E_0(\omega)e^{i\omega t} \qquad (62)$$



(c) Terms for j-th Stratum

(a) Multi-Layered Strata

(b) Outcrop of Eng. Bedrock

The design basis earthquake acceleration time history:

$$2\ddot{e}(t) \qquad (63)$$

is provided at the outcrop of the engineering bedrock, which means the engineering bedrock without the overlying soil strata(Surface strata).

Calculating the Fourier transform of the acceleration time history wave, the wave can be expressed by the superposition type of each frequency components:

$$2\ddot{e}(t) = \sum_{s} 2A(\omega_s)e^{i\omega_s t} \qquad (64)$$

where, $2A(\omega_s)$ indicates the Fourier amplitude of the design basis acceleration (Acceleration Fourier spectrum).

The double integral of Eq.(64) by the time t gives the displacement time history:

$$2e(t) = \sum_s [2A(\omega_s)/(i\omega_s)^2] e^{i\omega_s t} = -\sum_s [2A(\omega_s)/\omega_s^2] e^{i\omega_s t}$$

$$= \sum_s 2E_0(\omega_s) e^{i\omega_s t} \qquad (65)$$

where,

$$E_0(\omega_s) = -A(\omega_s)/\omega_s^2 \qquad (66)$$

Using Eq.(61):

$$\tilde{U}_j(\omega)\cdot 2E_0(\omega)e^{i\omega t} \qquad (61)$$

the displacement response time history at the j-th interface is given by:

$$u_j(t)=\sum_s \tilde{U}_j(\omega_s)\cdot 2E_0(\omega_s)e^{i\omega_s t} \qquad (67)$$

And, the acceleration response time history is also given by:

$$a_j(t)=\frac{d^2u_j(t)}{dt^2}=\sum_s \tilde{U}_j(\omega_s)\cdot 2E_0(\omega_s)\cdot\frac{d^2e^{i\omega_s t}}{dt^2}$$

$$=\sum_s \tilde{U}_j(\omega_s)\cdot 2E_0(\omega_s)\cdot(-\omega_s^2)e^{i\omega_s t}$$

$$=\sum_s \tilde{U}_j(\omega_s)\cdot 2A(\omega_s)e^{i\omega_s t} \qquad (68)$$

Furthermore, the shear strain time history $\gamma_j(t)$ in the j-th stratum is given by:

$$\gamma_j(t) = \sum_s \tilde{\Gamma}_j(\omega_s) \cdot 2E_0(\omega_s) e^{i\omega_s t}$$

$$= -\sum_s \tilde{\Gamma}_j(\omega_s) \cdot 2A(\omega_s) / \omega_s^2 e^{i\omega_s t} \qquad (69)$$

The calculations of Eq.(68) and Eq.(69) gives the response of acceleration and the shear strain.

$$a_j(t) = \sum_s \tilde{U}_j(\omega_s) \cdot 2A(\omega_s) e^{i\omega_s t} \qquad (68)$$

The computations for Eq.(68) and Eq.(69) are performed using FFT.

(5) Nonlinear Analysis

One of the numerical procedures for nonlinear analysis is the equivalent linear analysis procedure, where the equivalent shear modulus G_{ej} and the equivalent damping factor h_{ej} are determined in correspondence to the equivalent shear strain γ_{ej}.

The iteration of the computation is continued until the stable values are attained.

The famous computer program of the equivalent linear analysis procedure for the response of the soil strata is the program "SHAKE". In the SHAKE, the equivalent shear strains γ_{ej} for the each stratum are evaluated by:

$$\gamma_{ej} = 0.65 \cdot \max . \gamma_j(t) \qquad (70)$$

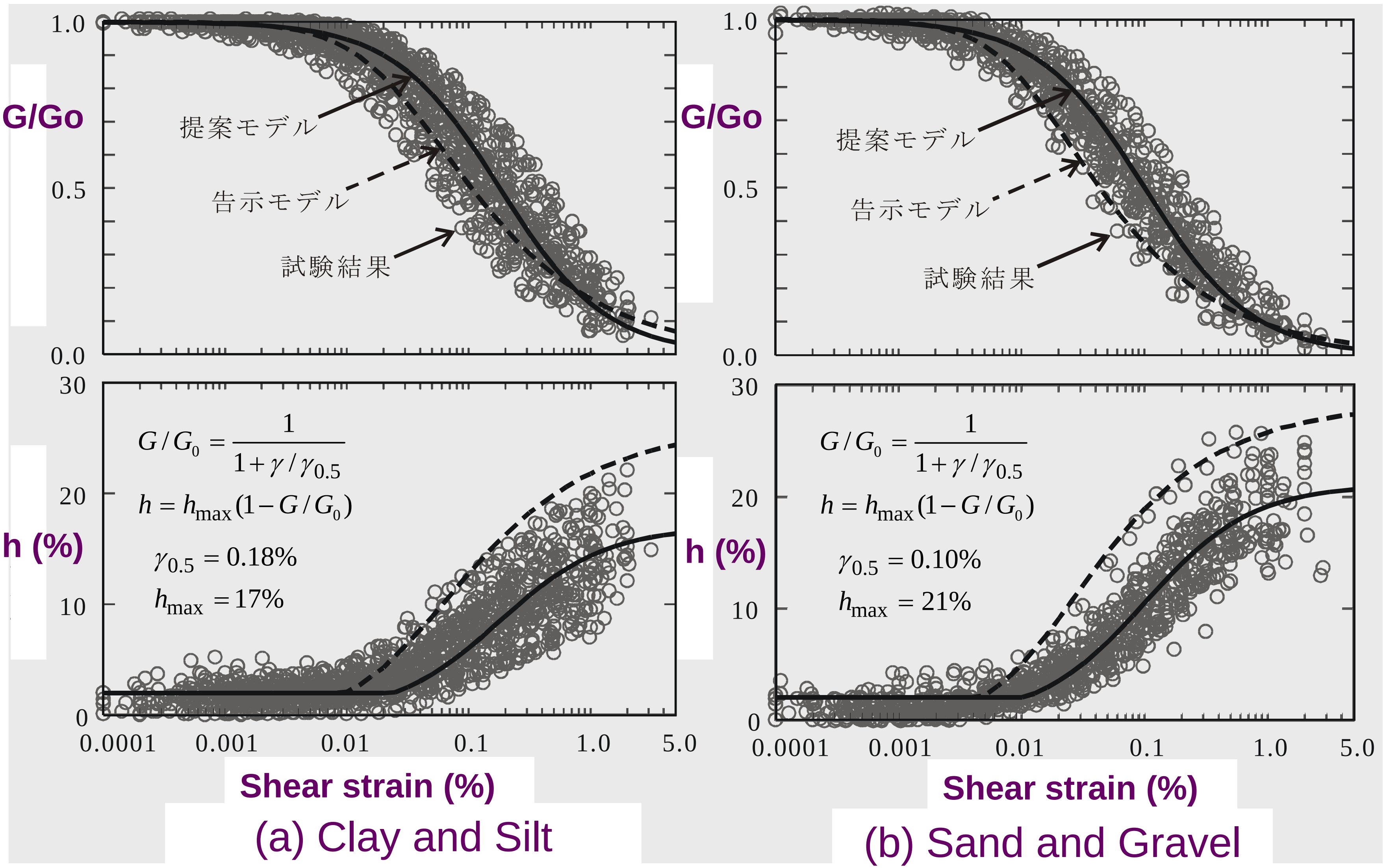
There are many nonlinear models which express the relation between the shear strain γ and the shear modulus G, the damping factor h.

Using the experimental test data on the soil specimens, the relations between γ and G/Go, h are regressed as:

$$G/Go=1/(1+\gamma/\gamma_{0.5}) \qquad h=h_{max}(1-G/Go)$$

Where, Go=initial shear modulus, h_{max} and $\gamma_{0.5}$ are given as followings.

- **Clay and Silt : h_{max}=17% , $\gamma_{0.5}$ =0.18%**
- **Sand and Gravel : h_{max}=21% , $\gamma_{0.5}$ =0.10%**


G/Go
1.0
0.5
0.0
提案モデル
告示モデル
試験結果
h (%)
30
20
10
0
$G/G_0 = \frac{1}{1+\gamma/\gamma_{0.5}}$
$h = h_{max}(1 - G/G_0)$
$\gamma_{0.5} = 0.18\%$
$h_{max} = 17\%$
$\gamma_{0.5} = 0.10\%$
$h_{max} = 21\%$
0.0001
0.001
0.01
0.1
1.0
5.0
Shear strain (%)
(a) Clay and Silt
(b) Sand and Gravel

E N D

One-Dimensional Shear Wave Propagation

In Multi-Layered Strata

(SHAKE)

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