Mathematics for Seismology

(Part 2)

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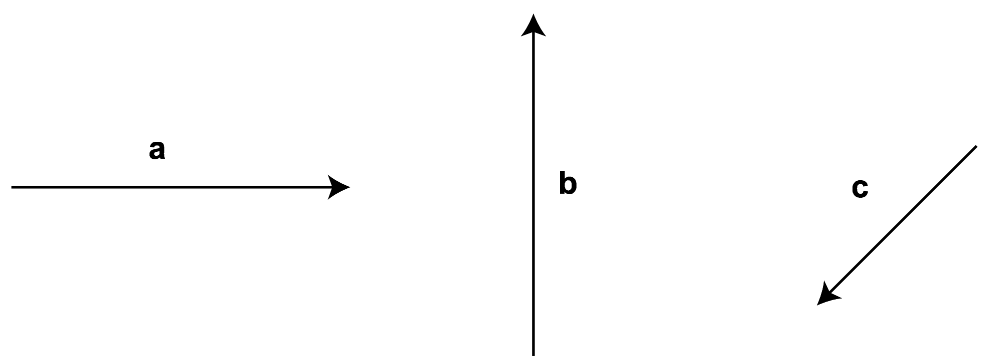
Chapter 1

Vector Algebra

1.1 Scalar, Vector and Tensor

Physical quantities are classified into the following three groups.
i) Scalar quantities are specified by their magnitudes only.
  # Examples —– mass m, length l, time t, temperature T, energy E
ii) Vector quantities are specified by their magnitudes and directions.
  # Examples —– displacement $\mathbf{u}$, velocity $\mathbf{v}$, acceleration $\mathbf{a}$, force $\mathbf{f}$, momentum $\mathbf{P}$
  A vector quantities are denoted by a directed line segment as:



A vector quantities are also denoted by their components as:

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

iii) Tensor quantities of order 2 or higher
  # Examples —– stress tensor $\boldsymbol{\sigma}= \sigma_{ij}$, strain tensor $\boldsymbol{\epsilon}= \epsilon_{ij}$ (order 2)
  elastic constant tensor C_{ijkl} (order 4)

1.2 Vector Algebra

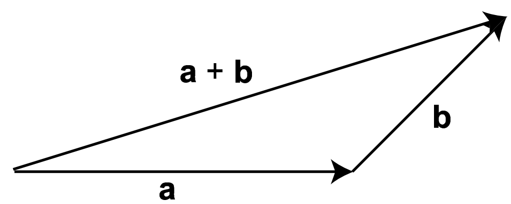
1.2.1 Vector Algebra in 2-D and 3-D space

(0) Length of Vector

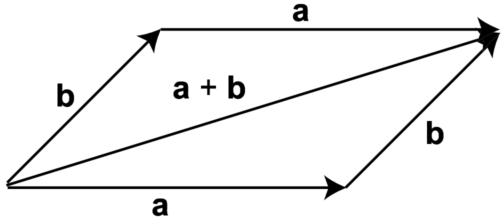
We denote the length of vector $\mathbf{a}$ as $|\mathbf{a}|$.

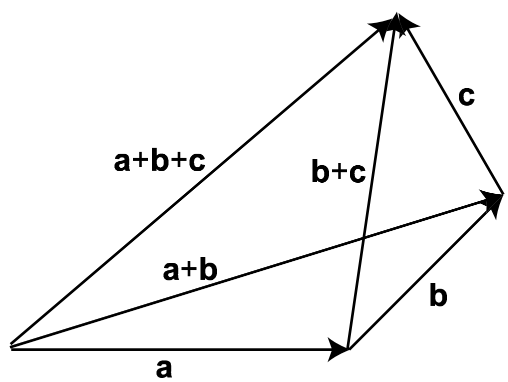
* Unit vector $\mathbf{n}$: $|\mathbf{n}| = 1$
* Zero vector $\mathbf{0}$: $|\mathbf{0}| = 0$

(1) Addition of Vectors

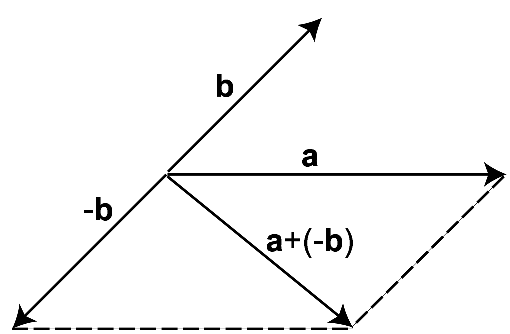


* Nature of Addition of Vectors
  Commutative law: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
  Associative law: $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$





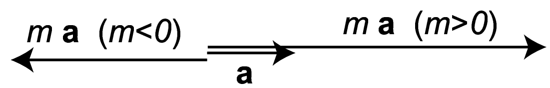
(2) Subtraction of Vectors
  $\mathbf{a}$-$\mathbf{b}$ is defined by $\mathbf{a}+(-\mathbf{b})$



(3) Multiplication of a Vector by a Scalar

Given a vector $\mathbf{a}$ and a scalar m, vector $m\mathbf{a}$ is defined as:

$|m\mathbf{a}| = |m||\mathbf{a}|$ and the direction is the same as $\mathbf{a}$ for $m > 0$
unspecified $\mathbf{a}$ for $m = 0$
the opposite to $\mathbf{a}$ for $m < 0$



* Nature of Multiplication of a Vector by a Scalar

Associative law: $m(n\mathbf{a}) = (mn)\mathbf{a}$

Distributive law: $(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

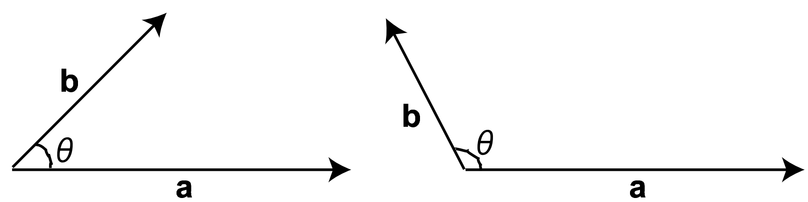
$m(\mathbf{a}+\mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

(4) Inner Product of Two Vectors

We denote inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$ as $\mathbf{a}\cdot\mathbf{b}$ or $(\mathbf{a},\mathbf{b})$, which is defined as:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



* Nature of Inner Product

Commutative law: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

Associative law: $(m\mathbf{a})\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b})$

Distributive law: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

* $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

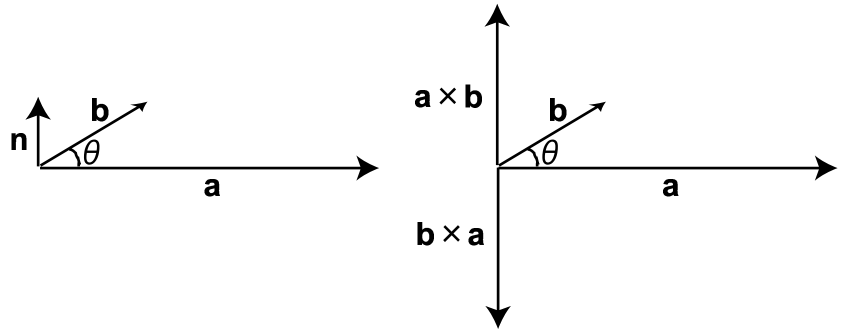
* $\mathbf{a}\cdot\mathbf{b} = 0 \Leftrightarrow |\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$ or $\mathbf{a}$ is perpendicular to $\mathbf{b}$

(5) Exterior Product of Vectors (defined only for 3-D space)

We denote exterior product of two vectors $\mathbf{a}$ and $\mathbf{b}$ as $\mathbf{a}\times\mathbf{b}$, which is defined as:

$$\mathbf{a}\times\mathbf{b} = (|\mathbf{a}|\ |\mathbf{b}|\ \sin\theta)\ \mathbf{n},$$

where $\mathbf{n}$ is the unit normal vector of the plane defined by $\mathbf{a}$ and $\mathbf{b}$, and $(\mathbf{a}\ \mathbf{b}\ \mathbf{n})$ is a right handed system.



* Not commutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
  (Associative law and distribute law still hold.)

* $\mathbf{a} \times \mathbf{a} = 0$

* $\mathbf{a} \times \mathbf{b} = 0 \Leftrightarrow \mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\mathbf{a}$ is parallel to $\mathbf{b}$

1.2.2 Vector Algebra in n-Dimensional Complex Space

Vectors in n-Dimensional Complex Space

Vectors in n-dimensional complex space:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

where a_1, a_2, $\cdots$, a_n are complex numbers.

* Length (norm) of vector $|\mathbf{a}|$ is defined as

$$|\mathbf{a}| = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}.$$

* Equality of two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} = \mathbf{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ \cdots,\ a_{n-1} = b_{n-1},\ \text{and } a_n = b_n$$

Vector Algebra

(1) Addition of Vectors

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

(2) Multiplication of a Vector by Scalar

$$m\mathbf{a} = \begin{pmatrix} ma_1 \\ ma_2 \\ \vdots \\ ma_n \end{pmatrix}$$

(3) Subtraction of Vectors

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{pmatrix}$$

(4) Inner Product of Two Vectors

$$\begin{aligned} (\mathbf{a}, \mathbf{b}) &= a_1 b_1^* + a_2 b_2^* + \cdots + a_n b_n^* \\ &= \sum_{i=1}^{n} a_i b_i^* \\ &= a_i b_i^* \qquad \text{(summation convention for subscripts)}, \end{aligned}$$

where b_i^* is the complex conjugate of b_i.

* If $\mathbf{x}$ and $\mathbf{y}$ satisfy $(\mathbf{x}, \mathbf{y}) = 0$, we say $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

* Nature of inner product

(1)

$$(\mathbf{y}, \mathbf{x}) = (\mathbf{x}, \mathbf{y})^*$$

(2)

$$(c\mathbf{x}, \mathbf{y}) = c\,(\mathbf{x}, \mathbf{y})\,, \quad (\mathbf{x}, c\mathbf{y}) = c^*\,(\mathbf{x}, \mathbf{y})$$

(3)

$$\begin{aligned} (\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) &= (\mathbf{x}_1, \mathbf{y}) + (\mathbf{x}_2, \mathbf{y}) \\ (\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) &= (\mathbf{x}, \mathbf{y}_1) + (\mathbf{x}, \mathbf{y}_2) \end{aligned}$$

Problems

Problem 1.1

Given $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$ are real vectors and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Confirm

$$|\mathbf{a}|\ |\mathbf{b}|\ \cos\theta = a_x b_x + a_y b_y + a_z b_z.$$

Problem 1.2

Given $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$ are real vectors, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is given as

$$\mathbf{c} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}.$$

Confirm

(i) $\mathbf{c}$ is orthogonal both to $\mathbf{a}$ and $\mathbf{b}$, and
(ii) $|\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$.

Problem 1.3

Show that

(1)

$$(\mathbf{y}, \mathbf{x}) = (\mathbf{x}, \mathbf{y})^*$$

(2)

$$(c\mathbf{x}, \mathbf{y}) = c\,(\mathbf{x}, \mathbf{y})\,, \qquad (\mathbf{x}, c\mathbf{y}) = c^*\,(\mathbf{x}, \mathbf{y})$$

(3)

$$(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) = (\mathbf{x}_1, \mathbf{y}) + (\mathbf{x}_2, \mathbf{y})$$
$$(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) = (\mathbf{x}, \mathbf{y}_1) + (\mathbf{x}, \mathbf{y}_2)$$

Problem 1.4

Show that

(1) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
(2) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Problem 1.5

Show that

(1) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$
(2) $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2\mathbf{a} \times \mathbf{b}$
(3) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

Chapter 2

Matrix Algebra

2.1 Notation of Matrix, Types of Matrix

Matrix is a set of numbers (real or complex) which are arranged in rows and columns.

Notation

We denote matrix as follows:

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

We call a_{ij} as the (i,j)-th element of matrix $\mathbf{A}$,
If a matrix has m rows and n columns, we say this matrix is order "m by n", which we denote as (m,n).

Transpose Matrix

Transpose matrix is the (n,m) matrix obtained by interchanging the rows and columns of an (m,n) matrix $\mathbf{A} = (a_{ij})$. We denote the transpose matrix of $\mathbf{A}$ as $\mathbf{A}^T$.

$$(a_{ij})^T = (a_{ji})$$

Adjoint Matrix

Transpose matrix is the (n,m) matrix obtained by converting each element to its complex conjugate and interchanging the rows and columns of an (m,n) matrix $\mathbf{A} = (a_{ij})$. We denote the adjoint matrix of $\mathbf{A}$ as $\mathbf{A}^*$.

$$(a_{ij})^* = \left(a_{ji}^*\right)$$

Zero Matrix

Zero matrix is a matrix whose elements are all zero.

ex)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Square, Diagonal and Unit Matrix

Square matrix is a matrix which has equal number of rows and columns. If square matrix has n rows and columns, we say it is order n square matrix.

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

We call elements a_{ii} $(1 \leq i \leq n)$ as diagonal elements.

$$\begin{aligned} \text{trace of } \mathbf{A} &= a_{11} + a_{22} + a_{33} + \cdots + a_{nn} \\ &= \sum_{i=1}^{n} a_{ii} \\ &= a_{ii} \qquad \text{(summation conventions)} \end{aligned}$$

Diagonal matrix is a matrix whose elements other than diagonal elements are all zero.

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

Unit matrix is a diagonal matrix whose diagonal elements are all 1. We usually denote unit matrix as **I**.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Symmetric matrix is a square matrix which satisfies

$$\mathbf{A}^T = \mathbf{A} \qquad \text{or } a_{ji} = a_{ij}.$$

Skew-symmetric matrix is a square matrix which satisfies

$$\mathbf{A}^T = -\mathbf{A} \qquad \text{or } a_{ji} = -a_{ij}.$$

Clearly $a_{ii} = 0$ for all i in a skew-symmetric matrix.
Hermite matrix is a square matrix which satisfies

$$\mathbf{A}^* = \mathbf{A} \qquad \text{or } a_{ji}^* = a_{ij}.$$

ex)

$$\begin{array}{cc} \text{Symmetric Matrix} & \text{Skew-symmetric Matrix} \\ S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix} & T = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix} \end{array}$$

* Arbitrary matrix can be decomposed to symmetric matrix and skew-symmetric matrix.

2.2 Matrix Algebra

2.2.1 Addition snd Subtraction

Addition and subtraction of matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as:

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$$

$$\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$$

Commutative law: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
Associative law: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

2.2.2 Scalar Multiplication

Scalar multiplication of matrix $\mathbf{A}$ is defined as:

$$m\mathbf{A} = (ma_{ij})$$

Distributive laws:
$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}$
$(m_1 + m_2)\,\mathbf{A} = m_1\mathbf{A} + m_2\mathbf{A}$

2.2.3 Multiplication of Matrices

We can define the multiplication of matrices **A** and **B** only when the number of columns of **A** and the number of rows of **B** are equal. The multiplication of (m,n) matrix **A** and (n,l) matrix **B** are defined as:

$$\begin{aligned}\mathbf{AB} &= \left(\sum_{k=1}^{m} a_{ik}b_{kj}\right) \qquad (1 \le i \le m, 1 \le j \le l)\\ &= (a_{ik}b_{kj}) \qquad \text{(summation convention)}\end{aligned}$$

The order of **AB** is (m,l).

ex)

(1)

$$\begin{pmatrix} 4 & 2 & -1 & 2 \\ 3 & -7 & 1 & -8 \\ 2 & 4 & -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -3 & 0 \\ 1 & 5 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (4\times 2)-(2\times 3)-(1\times 1)+(2\times 3) & (4\times 3)+(2\times 0)-(1\times 5)+(2\times 1) \\ (3\times 2)+(7\times 3)+(1\times 1)-(8\times 3) & (3\times 3)-(7\times 0)+(1\times 5)-(8\times 1) \\ (2\times 2)-(4\times 3)-(3\times 1)+(1\times 3) & (2\times 3)+(4\times 0)-(3\times 5)+(1\times 1) \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 9 \\ 4 & 6 \\ -8 & -8 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\begin{pmatrix} 5 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 10 & 4 & -6 \\ -5 & -2 & 3 \\ 20 & 8 & -12 \end{pmatrix}$$

(3)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

Thus the system of linear algebraic equation

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

can be expressed as the following matrix equation:

$$\mathbf{Ax} = \mathbf{y}$$

(4)

$$\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 26 \\ -4 & -9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix}$$

Thus commutative law does not hold for matrix multiplications.

(5)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 & 2 \\ -2 & -1 & -1 \\ -1 & -3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\mathbf{AB} = 0$ does not necessarily imply $\mathbf{A} = 0$ or $\mathbf{B} = 0$.

(6)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ -2 & -1 & -3 \\ -3 & -5 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Compareing examples (5) and (6), we can see $\mathbf{AB} = \mathbf{AC}$ does not necessarily imply $\mathbf{B} = \mathbf{C}$.

(7)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

In general, $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.

* Nature of Matrix Multiplication

Distributive law: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

Associative law: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

2.3 Determinant of Square Matrix

We write the determinant of square Matrix $\mathbf{A}$ as $|\mathbf{A}|$. Determinant is defined using permutation.

2.3.1 Permutation

Permutation is one to one mapping of a set of elements $\{1, 2, 3, \cdots, n\}$.
When σ is the permutation such as

$$\sigma(1) = i_1,\ \sigma(2) = i_2,\ \sigma(3) = i_3\ \cdots \sigma(n) = i_n$$

we write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}$$

* equal permutations:

ex)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

* even (odd) permutation:
  permutation which can be expressed by even (odd) times simple transactions of adjacent elements

2.3.2 Definition of Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$S_n: \text{ set of permutations for } n \text{ elements}$$
$$sgn(\sigma) = \begin{cases} 1 & \sigma \text{ is even permutation} \\ -1 & \sigma \text{ is odd permutation} \end{cases}$$

* Determinant of (1,1) matrix

$\sigma \in S_1$	$sgn(\sigma)$	$sgn(\sigma)a_{1\sigma(1)}$
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	even	$+a_{11}$

Thus

$$|a_{11}| = a_{11}$$

* Determinant of (2,2) matrix

$\sigma \in S_2$	$sgn(\sigma)$	$sgn(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}$
$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$	even	$+a_{11}a_{22}$
$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	odd	$-a_{12}a_{21}$

Thus

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

* Determinant of (3,3) matrix

$\sigma \in S_3$	$sgn(\sigma)$	$sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	even	$+a_{11}a_{22}a_{33}$
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	odd	$-a_{11}a_{23}a_{32}$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	odd	$-a_{12}a_{21}a_{33}$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	even	$+a_{12}a_{23}a_{31}$
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	even	$+a_{13}a_{21}a_{32}$
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	odd	$-a_{13}a_{22}a_{31}$

Thus

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\end{aligned}$$

2.3.3 Nature of Determinant

(1) $\left|\mathbf{A}^T\right| = |\mathbf{A}|$

(2) $|\mathbf{AB}| = |\mathbf{A}|\ |\mathbf{B}|$

(3)

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + \cdots + a_{2n}C_{2n} \\ &= \vdots \\ &= a_{n1}C_{n1} + a_{n2}C_{n2} + a_{n3}C_{n3} + \cdots + a_{nn}C_{nn}\end{aligned}$$

C_{ij} is (i, j)-th cofactor of $\mathbf{A}$:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

M_{ij} is (i,j)-th minor of $\mathbf{A}$:

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

ex) determinant of (2,2) matrix

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}\,|a_{22}| - a_{12}\,|a_{12}| \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

ex) determinant of (3,3) matrix

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) \\ &\quad + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ &\quad - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

(4)

$$\begin{aligned} &\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{(j-1)} & \mathbf{a}'_j + \mathbf{a}''_j & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{(j-1)} & \mathbf{a}'_j & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\ &\quad + \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{(j-1)} & \mathbf{a}''_j & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \end{aligned}$$

(5)

$$\begin{aligned} &\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{(j-1)} & c\,\mathbf{a}_j & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\ &= c\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{(j-1)} & \mathbf{a}_j & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \end{aligned}$$

(6)

$$\begin{vmatrix} \mathbf{a}_{\sigma(1)} & \mathbf{a}_{\sigma(2)} & \mathbf{a}_{\sigma(3)} & \cdots & \mathbf{a}_{\sigma(n)} \end{vmatrix} = sgn(\sigma)\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_n \end{vmatrix}$$

$(6)'$

$$\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_j & \cdots & \mathbf{a}_j & \cdots & \mathbf{a}_n \end{vmatrix} = 0$$

(6)″

$$\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{i-1} & \mathbf{a}_i + c\mathbf{a}_j & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_j & \cdots & \mathbf{a}_n \end{vmatrix}$$
$$= \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{i-1} & \mathbf{a}_i & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_j & \cdots & \mathbf{a}_n \end{vmatrix}$$

2.3.4 Computation of (n,n) Matrix Determinant

(3), (6) and (6)″ in the previous section are often used to compute N×N matrix determinant.

ex)

$$\begin{vmatrix} 3 & 1 & 2 & -3 \\ -2 & 3 & -5 & 2 \\ 5 & 2 & -1 & 3 \\ 1 & 5 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 3-3\times 1 & 1 & 2-2\times 1 & -3+3\times 1 \\ -2-3\times 3 & 3 & -5-2\times 3 & 2+3\times 3 \\ 5-3\times 2 & 2 & -1-2\times 2 & 3+3\times 2 \\ 1-3\times 5 & 5 & 4-2\times 5 & 2+3\times 5 \end{vmatrix}$$
$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -11 & 3 & -11 & 11 \\ -1 & 2 & -5 & 9 \\ -14 & 5 & -6 & 17 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & -11 & -11 & 11 \\ 2 & -1 & -5 & 9 \\ 5 & -14 & -6 & 17 \end{vmatrix}$$
$$= -\begin{vmatrix} -11 & -11 & 11 \\ -1 & -5 & 9 \\ -14 & -6 & 17 \end{vmatrix} = -836$$

2.4 Adjugate Matrix

Adjugate matrix of (n,n) matrix $\mathbf{A}$ is defined as:

$$\mathbf{A}^{\text{adj}} = \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & \ddots & \ddots & C_{n3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & \cdots & C_{nn} \end{pmatrix}$$

or

$$A^{\text{adj}}_{ij} = C_{ji},$$

where C_{ij} is (i,j)-th cofactor of $\mathbf{A}$.

ex) $\mathbf{A} = \begin{pmatrix} 2 & 0 & 7 \\ -1 & 4 & 5 \\ 3 & 1 & 2 \end{pmatrix}$,

$$\begin{aligned}
\mathbf{A}^{\text{adj}} &= \begin{pmatrix} \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 7 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 7 \\ 4 & 5 \end{vmatrix} \\ -\begin{vmatrix} -1 & 5 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 7 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 7 \\ -1 & 5 \end{vmatrix} \\ \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 3 & 7 & -28 \\ 17 & -17 & -17 \\ -13 & -2 & 8 \end{pmatrix}
\end{aligned}$$

* Adjugate matrix satisfies following relation:

$$\mathbf{A}^{\text{adj}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\text{adj}} = |\mathbf{A}|\ \mathbf{I}$$

For (3,3) matrix,

$$\begin{aligned}
\mathbf{A}^{\text{adj}}\mathbf{A} &= \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} & a_{12}C_{11} + a_{22}C_{21} + a_{32}C_{31} & a_{13}C_{11} + a_{23}C_{21} + a_{33}C_{31} \\ a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32} & a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} & a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32} \\ a_{11}C_{13} + a_{21}C_{23} + a_{31}C_{33} & a_{12}C_{13} + a_{22}C_{23} + a_{32}C_{33} & a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{pmatrix} \\
&= \begin{pmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{12} & a_{13} \\ a_{23} & a_{22} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} & a_{13} \\ a_{21} & a_{23} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{11} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{pmatrix} \\
&= \begin{pmatrix} |\mathbf{A}| & 0 & 0 \\ 0 & |\mathbf{A}| & 0 \\ 0 & 0 & |\mathbf{A}| \end{pmatrix} = |\mathbf{A}|\,\mathbf{I}
\end{aligned}$$

2.5 Inverse Matrix

Inverse matrix of (n,n) matrix $\mathbf{A}$ is defined as:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^{\text{adj}}$$

This matrix satisfies:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

* Nature of Inverse Matrix

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

2.6 Singular Matrix

For (n,n) matrix $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$, if $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ are linearly dependent, we call matrix $\mathbf{A}$ is singular.

* Linearly Independent:
When vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ satisfies

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = 0$$

only for $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}$ (we say these vectors have trivial linear relation only), we call $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ are linearly independent.

* Linearly dependent:
When vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ satisfies

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = 0$$

for some $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \neq \mathbf{0}$ (we say these vectors have non-trivial linear relation), we call $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ are linearly dependent.

* Determinant of a singular matrix is 0.

For a singular matrix $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$, we have the relation

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = 0$$

for some $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \neq \mathbf{0}$. If $c_j \neq 0$, we have

$$\mathbf{a}_j = -\frac{c_1}{c_j}\mathbf{a}_1 - \frac{c_2}{c_j}\mathbf{a}_2 - \cdots - \frac{c_{j-1}}{c_j}\mathbf{a}_{j-1} - \frac{c_{j+1}}{c_j}\mathbf{a}_{j+1} - \cdots - \frac{c_n}{c_j}\mathbf{a}_n$$

Determinant of matrix $\mathbf{A}$ is evaluated as follows:

$$\begin{aligned}
& \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\
= & -\frac{c_1}{c_j}\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{j-1} & \mathbf{a}_1 & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\
& \qquad\qquad\vdots \\
& -\frac{c_{j-1}}{c_j}\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{j-1} & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\
& -\frac{c_{j+1}}{c_j}\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\
& \qquad\qquad\vdots \\
& -\frac{c_n}{c_j}\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{j-1} & \mathbf{a}_n & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{vmatrix} \\
= & \ 0
\end{aligned}$$

Problems

Problem 2.1

(1)

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -3 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2.2

Show that

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$$

if and only if $\mathbf{A}$ and $\mathbf{B}$ commute,

$$\mathbf{AB} = \mathbf{BA}$$

Problem 2.3

Given

$$\mathbf{K} = \begin{pmatrix} 0 & 0 & i \\ -i & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

show that

$$\mathbf{K}^n = \mathbf{I}$$

with the proper choice of n ($n \neq 0$).

Problem 2.4

Show that

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{C}]] - [\mathbf{C}, [\mathbf{A}, \mathbf{B}]],$$

where

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$$

Problems 2.5

Compute the determinant.

(1) $\begin{vmatrix} 3 & -5 & 2 \\ 2 & 0 & 1 \\ -2 & 3 & 5 \end{vmatrix}$ (2) $\begin{vmatrix} 3 & 2 & 5 & -4 \\ -7 & 1 & -8 & 6 \\ 10 & 3 & 6 & 1 \\ 2 & 5 & 4 & 3 \end{vmatrix}$

Problem 2.6

Compute the inverse matrix.

(1) $\begin{pmatrix} 3 & -5 & 2 \\ 2 & 0 & 1 \\ -2 & 3 & 5 \end{pmatrix}$ (2) $\begin{pmatrix} 3 & 2 & 5 & -4 \\ -7 & 1 & -8 & 6 \\ 10 & 3 & 6 & 1 \\ 2 & 5 & 4 & 3 \end{pmatrix}$

Problem 2.7

Show that

$$\begin{vmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{vmatrix} = a_1 a_2 \cdots a_n$$

Problem 2.8

Show that

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & c & b \\ -b & -c & 0 & a \\ -c & -b & -a & 0 \end{vmatrix} = \left(a^2 - b^2 + c^2\right)^2$$

Chapter 3

Eigenvalue Problem

3.1 Eigenvalue and Eigenvector

If square matrix $\mathbf{A}$ satisfies the equation

$$\mathbf{A}\mathbf{x} = \alpha\mathbf{x}$$

for some $\mathbf{x} \neq \mathbf{0}$, we call α as eigenvalue and $\mathbf{x}$ as corresponding eigenvector.

3.1.1 How to Find Eigenvalues and Eigenvectors

Eigenvalues:

$$\begin{aligned}\mathbf{A}\mathbf{x} = \alpha\mathbf{x} \quad &\Leftrightarrow \quad (\mathbf{A} - \alpha\mathbf{I})\,\mathbf{x} = \mathbf{0} \\ &\Leftrightarrow \quad |\mathbf{A} - \alpha\mathbf{I}| = 0\end{aligned}$$

Solving the above equation, we can obtain eigenvalues α.

Eigenvectors:
If

$$\mathbf{A} - \alpha\mathbf{I} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix},$$

we can expect some non-trivial linear relations:

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = 0.$$

Then $\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is eigenvector for eigenvalue α.

Note: $\mathbf{x}$ is not unique for each α. Sometimes we can find linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m$ for one eigenvalue.

ex)

1. Eigenvalues and eigenvectors for $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

$$|\mathbf{A} - \alpha\mathbf{I}| = \begin{vmatrix} 1-\alpha & 1 \\ 0 & 2-\alpha \end{vmatrix} = (1-\alpha)(2-\alpha)$$

Thus

$$|\mathbf{A} - \alpha\mathbf{I}| = 0 \Leftrightarrow \boxed{\alpha = 1, 2}$$

$$\mathbf{A} - 1\,\mathbf{I} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{thus } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is eigenvector for } \alpha = 1$$

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{thus } \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is eigenvector for } \alpha = 2$$

2. Eigenvalues and eigenvectors for $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

$$|\mathbf{A} - \alpha\mathbf{I}| = \begin{vmatrix} 1-\alpha & 2 \\ 0 & 1 \end{vmatrix} = (1-\alpha)^2$$

Thus

$$|\mathbf{A} - \alpha\mathbf{I}| = 0 \Leftrightarrow \boxed{\alpha = 1}$$

$$\mathbf{A} - 1\,\mathbf{I} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \text{thus } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is eigenvector for } \alpha = 1$$

3.2 Eigenvalues of Hermite Matrix

Hermite Matrix:

$$\mathbf{A}^* = \mathbf{A} \qquad \text{or} \qquad a_{ji}^* = a_{ij}$$

Real Hermite Matrix = Symmetric Matrix.

$$\mathbf{A}^T = \mathbf{A} \qquad \text{or} \qquad a_{ji} = a_{ij}$$

Eigenvalues and eigenvectors of Hermite matrix have following characters:

1. Eigenvalues of Hermite matrix are all real.
2. Eigenvalues of Hermite matrix are mutually orthogonal

proof)

1. Let α and $\mathbf{x}$ are an eigenvalue and its corresponding eigenvector of Hermite matrix $\mathbf{A}$.

$$\mathbf{A}\mathbf{x} = \alpha\mathbf{x}$$

Multiplying $\mathbf{x}^*$ from left, we obtain

$$\mathbf{x}^*\mathbf{A}\mathbf{x} = \alpha\mathbf{x}^*\mathbf{x} \tag{3.1}$$

On the other hand, because $\mathbf{A}^* = \mathbf{A}$, we obtain

$$\mathbf{A}^*\mathbf{x} = \alpha\mathbf{x}$$

Taking complex conjugate of both hand sides, we obtain

$$\mathbf{x}^*\mathbf{A} = \alpha^*\mathbf{x}^*$$

Multiplying $\mathbf{x}$ from right, we have

$$\mathbf{x}^*\mathbf{A}\mathbf{x} = \alpha\mathbf{x}^*\mathbf{x} \tag{3.2}$$

Comparing eqs. (3.1) and (3.2), we obtain

$$\alpha = \alpha^*$$

Thus eigenvectors of Hermite matrix is real.

2. This is very important theorem. But proof is difficult. Consult the references.

Problems

Problem 3.1

Find eigenvalues and corresponding eigenvectors.

(1) $\begin{pmatrix} -5 & 6 & 4 \\ -7 & 8 & 4 \\ -2 & 2 & 3 \end{pmatrix}$ (2) $\begin{pmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}$ (3) $\begin{pmatrix} 5 & 0 & -6 \\ 3 & -1 & -3 \\ 3 & 0 & -4 \end{pmatrix}$

Problem 3.2

Find eigenvales and corresponding eigenvecors for the following Hermite matrix.

$$\begin{pmatrix} 0 & i & 1 \\ -i & 0 & i \\ 1 & -i & 0 \end{pmatrix}$$

Chapter 4

Diagonalization

4.1 Diagonalization

Diagonalization of (n,n) square matrix $\mathbf{A}$ is to find a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

Note: This is not always possible.

4.1.1 How to Diagonalize a Square Matrix

Considering the case that (n,n) matrix $\mathbf{A}$ has n different (linearly independent) eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\cdots$, $\mathbf{x}_n$ eigenvalues. If we choose

$$\mathbf{P} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix},$$

$\mathbf{A}$ is diagonalized as:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}$$

where α_1, α_2, $\cdots$, α_n are corresponding eigenvalues.
proof)

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{P} &= \mathbf{P}^{-1}\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{P}^{-1}\mathbf{x}_1 & \mathbf{P}^{-1}\mathbf{x}_2 & \cdots & \mathbf{P}^{-1}\mathbf{x}_n \end{pmatrix} \end{aligned} \tag{4.1}$$

On the other hand,

$$\mathbf{P}^{-1}\mathbf{P} = \mathbf{I} \tag{4.2}$$

Comparing eqs. (4.1) and (4.2), we have

$$\mathbf{P}^{-1}\mathbf{x}_i = \mathbf{e}_i,$$

where

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{—— } i\text{-th component}$$

Thus $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ can be evaluated as follows:

$$\begin{aligned}
\mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \mathbf{P}^{-1}\mathbf{A}\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix} \\
&= \mathbf{P}^{-1}\begin{pmatrix} \alpha_1\mathbf{x}_1 & \alpha_2\mathbf{x}_2 & \cdots & \alpha_n\mathbf{x}_n \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1\mathbf{P}^{-1}\mathbf{x}_1 & \alpha_2\mathbf{P}^{-1}\mathbf{x}_2 & \cdots & \alpha_n\mathbf{P}^{-1}\mathbf{x}_n \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1\mathbf{e}_1 & \alpha_2\mathbf{e}_2 & \cdots & \alpha_n\mathbf{e}_n \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}
\end{aligned}$$

ex)

1) Diagonalization of $\mathbf{A} = \begin{pmatrix} 6 & -3 & -7 \\ -1 & 2 & 1 \\ 5 & -3 & -6 \end{pmatrix}$.

$\alpha = 1, 2, -1$ are eigenvalues and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are corresponding eigenvectors. Thus

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ for } \mathbf{P} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

2) Diagonalization of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix}$.

$\alpha = 2, 2, 1$ are eigenvalues and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ are corresponding eigenvectors. Thus

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } \mathbf{P} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 3 & 2 \end{pmatrix}$$

4.2 Diagonalization of Hermite Matrix

Hermite Matrix:

$$\mathbf{A}^* = \mathbf{A} \qquad \text{or} \qquad a_{ji}^* = a_{ij}$$

Real Hermite Matrix = Symmetric Matrix:

$$\mathbf{A}^T = \mathbf{A} \qquad \text{or} \qquad a_{ji} = a_{ij}$$

Unitary Matrix:

$$\mathbf{A}^* = \mathbf{A}^{-1}$$

Real Unitary Matrix = Orthogonal Matrix:

$$\mathbf{A}^T = \mathbf{A}^{-1}$$

* Hermite matrix can be diagonalized by Unitary Matrix.

Problems

Problem 4.1

Diagonalize the following matrices if possible.

$$(1)\begin{pmatrix} -5 & 6 & 4 \\ -7 & 8 & 4 \\ -2 & 2 & 3 \end{pmatrix} \qquad (2)\begin{pmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} \qquad (3)\begin{pmatrix} 5 & 0 & -6 \\ 3 & -1 & -3 \\ 3 & 0 & -4 \end{pmatrix}$$

Problem 4.2

Diagonalize the following Hermite matrix.

$$\begin{pmatrix} 0 & i & 1 \\ -i & 0 & i \\ 1 & -i & 0 \end{pmatrix}$$

Chapter 5

Vector Analysis

5.1 Differentiation of Vector Functions

Suppose $\mathbf{a} = (a_x, a_y, a_z)$ is a function of t, the differentiation of $\mathbf{a}$ w.r.t. t is defined as follows:

$$\frac{d\mathbf{a}}{dt} = \begin{pmatrix} \frac{da_x}{dt} \\ \frac{da_y}{dt} \\ \frac{da_z}{dt} \end{pmatrix}.$$

- $\dfrac{d}{dt}(\phi\mathbf{a}) = \phi\dfrac{d\mathbf{a}}{dt} + \dfrac{d\phi}{dt}\mathbf{a}$, where $\phi(t)$ is a scalar function.
- $\dfrac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt}\cdot\mathbf{b}$
- $\dfrac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\dfrac{d\mathbf{b}}{dt} + \dfrac{d\mathbf{a}}{dt}\times\mathbf{b}$

5.2 Vector Field

Vector function $\mathbf{a}$ as a function of space (x, y, z): vector field $\mathbf{a}(x, y, z)$.

5.3 Vector Operators

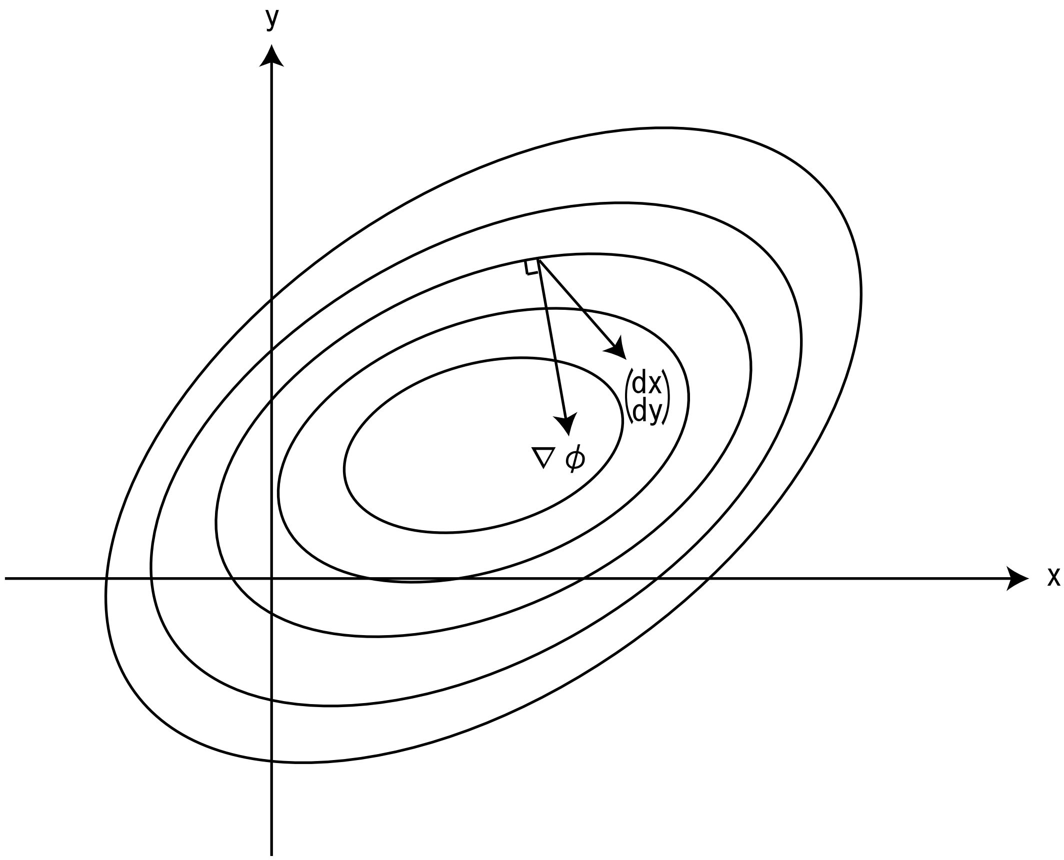
5.3.1 Gradient

For a scalar field $\phi(x, y, z)$,

$$\operatorname{grad}\phi = \nabla\phi = \begin{pmatrix} \dfrac{\partial\phi}{\partial x} \\ \dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial\phi}{\partial z} \end{pmatrix}.$$

* Physical interpretation

$\nabla\phi$ is a vector having the direction of the maximum space rate of change of ϕ.



$$
\begin{aligned}
d\phi &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \\
&= \nabla\phi \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} \\
&= \nabla\phi \sqrt{dx^2 + dy^2} \cos\theta,
\end{aligned}
$$

where θ is the angle between $\nabla\phi$ and $\begin{pmatrix} dx \\ dy \end{pmatrix}$.

For $\begin{pmatrix} dx \\ dy \end{pmatrix}$ $(\neq \mathbf{0})$,

$$
\frac{d\phi}{\sqrt{dx^2 + dy^2}} \leq \nabla\phi.
$$

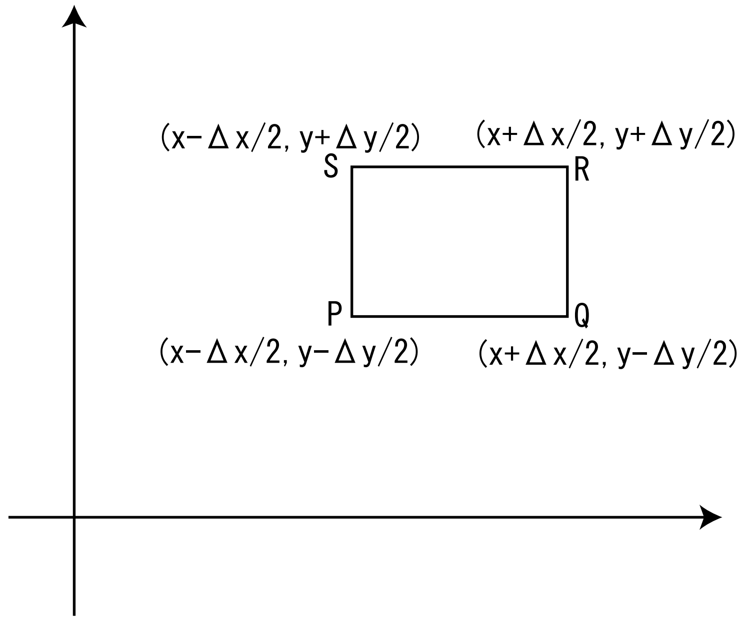
Thus $\nabla\phi$ is the maximum space rate change of ϕ.

5.3.2 Divergence

For a vector field $\mathbf{a}(x, y, z)$,

$$
\operatorname{div} \mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.
$$

* Physical Interpretation



Consider a flux field $\mathbf{v}(x, y)$.
Flow out at QR:

$$v_x(x + \Delta x/2, y)\Delta y$$

Flow in at PS:

$$v_x(x - \Delta x/2, y)\Delta y$$

Flow out at SR:

$$v_y(x, y + \Delta y/2)\Delta x$$

Flow in at PQ:

$$v_y(x, y - \Delta y/2)\Delta x$$

Net flow out of PQRS:

$$\begin{aligned} & v_x(x + \Delta x/2, y)\Delta y - v_x(x - \Delta x/2, y)\Delta y \\ & \quad + v_y(x, y + \Delta y/2)\Delta x - v_y(x, y - \Delta y/2)\Delta x \\ = & \ \frac{\partial v_x}{\partial x}(x, y)\Delta x\Delta y + \frac{\partial v_y}{\partial y}(x, y)\Delta x\Delta y \\ = & \left(\frac{\partial v_x}{\partial x}(x, y) + \frac{\partial v_y}{\partial y}(x, y)\right)\Delta x\Delta y \\ = & \ \nabla \cdot \mathbf{v}\ \Delta x\Delta y \end{aligned}$$

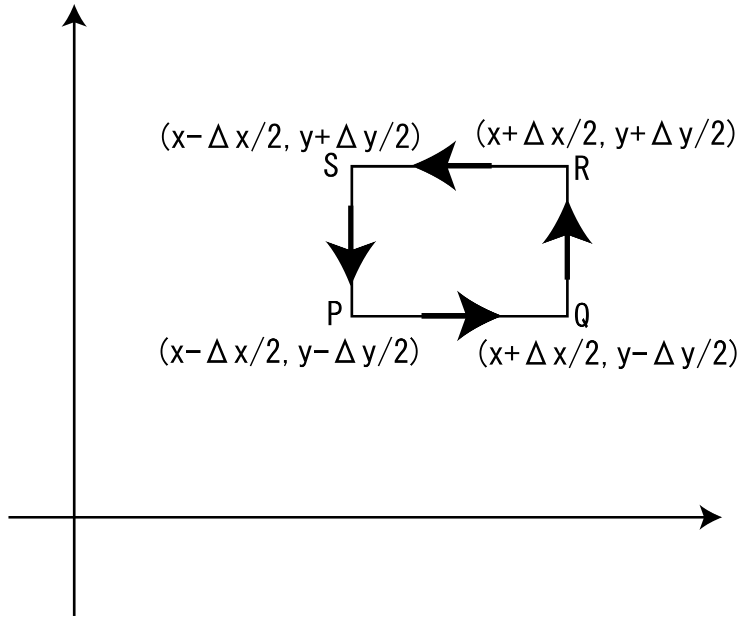
Thus, $\nabla \cdot v$ is a net flow out per unit volume.

5.3.3 Rotation

For a vector field $\mathbf{a}(x, y, z)$,

$$\mathrm{rot}\,\mathbf{a} = \nabla \times \mathbf{a} = \begin{pmatrix} \dfrac{\partial a_z}{\partial y} - \dfrac{\partial a_y}{\partial z} \\ \dfrac{\partial a_x}{\partial z} - \dfrac{\partial a_z}{\partial x} \\ \dfrac{\partial a_y}{\partial x} - \dfrac{\partial a_x}{\partial y} \end{pmatrix}.$$

* Physical Interpretation



Consider a flux field $\mathbf{v}(x, y, z)$.
Circulation in (x, y)-plane is as follows:

$$\text{circulation} = \int_{PQ} \mathbf{v} \cdot d\mathbf{s} + \int_{QR} \mathbf{v} \cdot d\mathbf{s} + \int_{RS} \mathbf{v} \cdot d\mathbf{s} + \int_{SP} \mathbf{v} \cdot d\mathbf{s}.$$

$$\begin{aligned}
\int_{QR} \mathbf{v} \cdot d\mathbf{s} &= v_y(x + \Delta x/2, y)\, \Delta y \\
\int_{SP} \mathbf{v} \cdot d\mathbf{s} &= -v_y(x - \Delta x/2, y)\, \Delta y \\
\int_{RS} \mathbf{v} \cdot d\mathbf{s} &= -v_x(x, y + \Delta y/2)\, \Delta x \\
\int_{PQ} \mathbf{v} \cdot d\mathbf{s} &= v_x(x, y - \Delta y/2)\, \Delta x
\end{aligned} \tag{5.1}$$

We obtain

$$\begin{aligned}
\text{circulation} &= v_y(x + \Delta x/2, y)\, \Delta y - v_y(x - \Delta x/2, y)\, \Delta y \\
&\quad - v_x(x, y + \Delta y/2)\, \Delta x + v_x(x, y - \Delta y/2)\, \Delta x \\
&= \frac{\partial v_y}{\partial x}(x, y)\, \Delta x \Delta y - \frac{\partial v_x}{\partial y}(x, y)\, \Delta x \Delta y \\
&= \left(\frac{\partial v_y}{\partial x}(x, y) - \frac{\partial v_x}{\partial y}(x, y) \right) \Delta x \Delta y.
\end{aligned}$$

Thus, z-component of $\nabla \times \mathbf{v}$ is a circulation in (x, y)-plane per unit area.

5.3.4 Laplacian

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

For a scalr field $\phi(x, y, z)$,

$$\nabla^2 \phi = \operatorname{div} \operatorname{grad} \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

For a vactor field $\mathbf{a}(x, y, z)$,

$$\nabla^2 \mathbf{a} = \begin{pmatrix} \dfrac{\partial^2 a_x}{\partial x^2} + \dfrac{\partial^2 a_x}{\partial y^2} + \dfrac{\partial^2 a_x}{\partial z^2} \\ \dfrac{\partial^2 a_y}{\partial x^2} + \dfrac{\partial^2 a_y}{\partial y^2} + \dfrac{\partial^2 a_y}{\partial z^2} \\ \dfrac{\partial^2 a_z}{\partial x^2} + \dfrac{\partial^2 a_z}{\partial y^2} + \dfrac{\partial^2 a_z}{\partial z^2} \end{pmatrix}.$$

5.4 Formulas of Vector Analysis

- $\nabla(\phi\,\psi) = \phi\nabla\psi + \psi\nabla\phi$
- $\nabla \cdot (\phi\,\mathbf{a}) = (\nabla\phi) \cdot \mathbf{a} + \phi\,(\nabla \cdot \mathbf{a})$
- $\nabla \times (\phi\,\mathbf{a}) = (\nabla\phi) \times \mathbf{a} + \phi\,(\nabla \times \mathbf{a})$
- $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$
- $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\,\mathbf{a} - \mathbf{b}\,(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\,\mathbf{b} + \mathbf{a}\,(\nabla \cdot \mathbf{b})$
- $\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \nabla)\,\mathbf{a} + (\mathbf{a} \cdot \nabla)\,\mathbf{b} + \mathbf{b} \times (\nabla \times \mathbf{a}) + \mathbf{a} \times (\nabla \times \mathbf{b})$
- $\nabla \times (\nabla\phi) = \operatorname{rot} \operatorname{grad} \phi = 0$
- $\nabla \cdot (\nabla \times \mathbf{a}) = \operatorname{div} \operatorname{rot} \mathbf{a} = 0$
- $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$

5.5 Potential Field

5.5.1 Scalar Potential

If the vector field $\mathbf{f}$ is expressed as

$$\mathbf{f} = \nabla\phi,$$

we call ϕ is a scalar potential of $\mathbf{f}$.

* $\nabla \times \mathbf{f} = \nabla \times (\nabla\phi) = 0$

5.5.2 Vector Potential

If the vector field $\mathbf{g}$ is expressed as

$$\mathbf{g} = \nabla \times \boldsymbol{\psi},$$

we call $\boldsymbol{\psi}$ is a vector potential of $\mathbf{g}$.

* $\nabla \cdot \mathbf{g} = \nabla \cdot (\nabla \times \boldsymbol{\psi}) = 0$

5.5.3 Helmholtz's Theorem

Arbitrary vector field $\mathbf{u}$ (strictly speaking, the vector field $\mathbf{u}$ with $\nabla \cdot \mathbf{u} = 0$ and $\nabla \times \mathbf{u} = 0$ at infinity) can be expressed as follows:

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}.$$

ex)
Elastic equation of motion:

$$(\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0.$$

Substituting the above expression, $\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}$, we obtain

$$(\lambda + 2\mu) \nabla \left(\nabla^2 \phi\right) - \mu \nabla \times (\nabla \times (\nabla \times \boldsymbol{\psi})) - \rho \frac{\partial^2}{\partial t^2} (\nabla \phi + \nabla \times \boldsymbol{\psi}) = 0.$$

Using

$$\nabla \times (\nabla \times \boldsymbol{\psi}) = \nabla (\nabla \cdot \boldsymbol{\psi}) - \nabla^2 \boldsymbol{\psi},$$

we have

$$\nabla \left[(\lambda + 2\mu) \nabla^2 \phi - \rho \frac{\partial^2 \phi}{\partial t^2}\right] + \nabla \times \left[\mu \nabla^2 \boldsymbol{\psi} - \rho \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}\right] = 0.$$

This reduces to the following two equations:

$$\begin{aligned} (\lambda + 2\mu) \nabla^2 \phi - \rho \frac{\partial^2 \phi}{\partial t^2} &= 0 \\ \mu \nabla^2 \boldsymbol{\psi} - \rho \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} &= 0. \end{aligned} \tag{5.2}$$

These equations represent P-wave and S-wave propagations, respectively.