3. Energy Transport and Group Velocity of Waves

3. 1. Wave Group Velocity

In the previous discussion, we proved that the phase velocity c of a wave is given by

$$c = \sigma / k = \sqrt{\frac{g}{k}\tanh kD} \qquad (3\text{-}1)$$

This is called "phase velocity."

Next, we consider the velocity of a propagating wave front into a motionless water area, i. e., an area where there is no wave motion.

First, we consider the one-dimensional problem (x-axis only) and two waves with a small wave number difference and having the same wave amplitude as

$$\eta_1 = a\sin(kx - \sigma t) \quad \text{and} \quad \eta_2 = a\sin\{(k+\Delta k)x - (\sigma + \Delta\sigma)t\} \qquad (3\text{-}2)$$

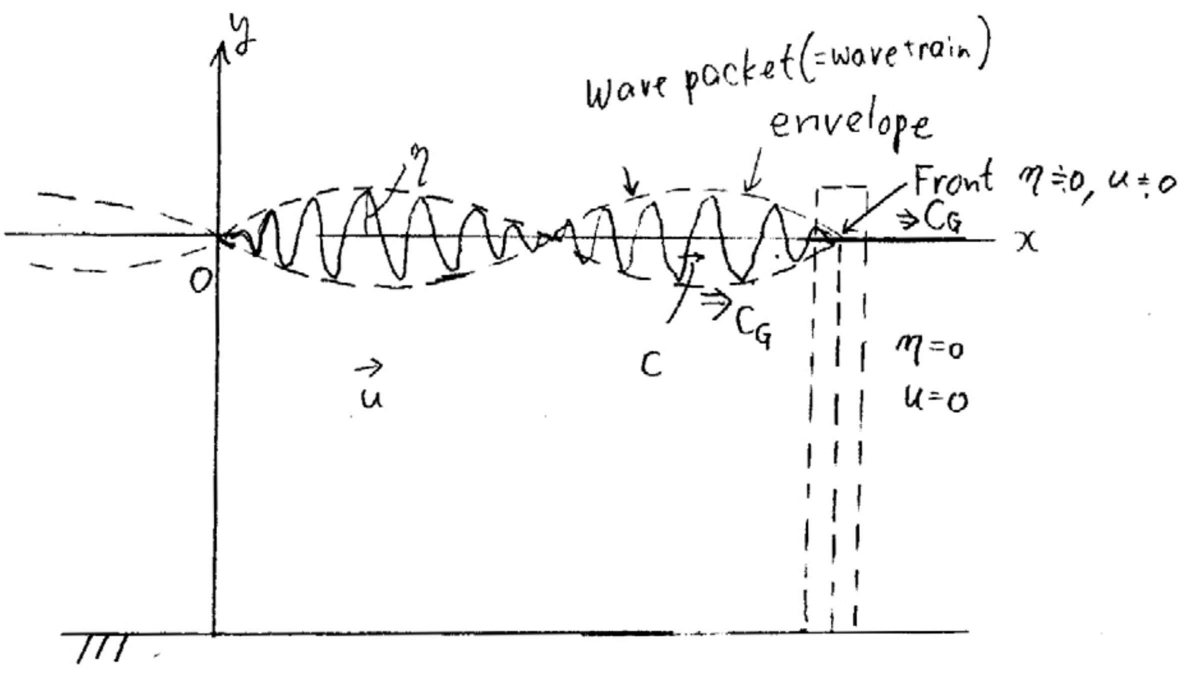

Fig. 3.1 Overlap of two waves creating wave packets

If the two waves exist simultaneously, the sea surface displacement is given by

$$
\begin{aligned}
\eta &= \eta_1 + \eta_2 = a\sin(kx - \sigma t) + a\sin\{(k+\Delta k)x - (\sigma + \Delta\sigma)t\} \\
&= 2a\cos(\Delta kx - \Delta\sigma t)\sin\left\{\frac{(2k+\Delta k)}{2}x - \frac{(2\sigma + \Delta\sigma)}{2}t\right\} \\
&\approx 2a\cos(\Delta kx - \Delta\sigma t)\sin(kx - \sigma t)
\end{aligned}
\tag{3.3}
$$

If we assume that Δk and $\Delta\sigma$ are satisfactorily small as compared with k and σ, $2a\cos(\Delta kx - \Delta\sigma t)$ changes slowly and is regarded as the "local amplitude."

Thus, (4.3) expresses a chain of wave packets (a wave group) having an envelope of $2a\cos(\Delta kx - \Delta\sigma t)$. The velocity of the wave group c_G is given by

$$c_G = \frac{\Delta\sigma}{\Delta k} \approx \frac{d\sigma}{dk} \tag{3.4}$$

In the case of an ocean wave, the dispersion relation is given by

$$\sigma^2 = gk\tanh kD \tag{3.5}$$

Hence, we differentiate this equation with respect to k, and we have

[Note: $y = \tanh x, y' = \sec h^2 x$]

$$c_G = \frac{1}{2}\left(1 + \frac{2kD}{\sinh 2kD}\right)\sqrt{\frac{g}{k}\tanh kD} = \frac{1}{2}\left(1 + \frac{2kD}{\sinh 2kD}\right)c \tag{3.6}$$

For the case of a deep water wave $(kD \to \infty)$, the group velocity becomes

$$c_G = \frac{1}{2}c \tag{3.7}$$

Thus, the velocity of the wave group is half that of the phase velocity, and hence, an individual wave appears at the rear of each wave packet, and disappears at its front.

For the case of a long wave, $(kD \to 0)$,

$$c_G = c \tag{3.8}$$

T he group velocity is equal to the phase velocity.

Since both η (sea surface displacement) and ϕ (velocity potential) become close to zero at the front (a node) of each package, we can replace in front of the front of a wave package into no motion area. Thus, a wave front spreads in the velocity of the group velocity. The total energy does not cross the nodal points, and hence, we can recognize that the wave energy is transported in the speed of the group velocity.