Theory of Seismic Waves

(Part 1)

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Chapter 1

Basic Elasticity Theory

1.1 Stress

Type of Forces

- body forces
  - forces proportional to the volume of the object
  - e.g.) gravity
- contact forces
  - forces proportional to surface area
  - e.g.) pressure in a fluid

Definition of Traction

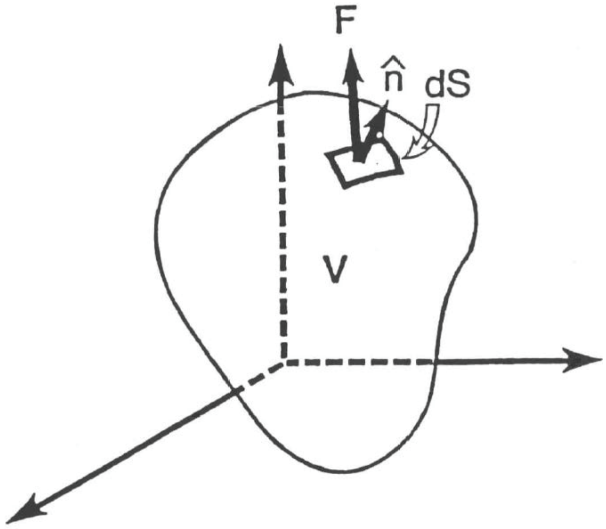


Figure 1.1 (from Geller, 1993)

– a small volume V within a larger continuous medium.
– surface force $\mathbf{F}$ acting on an element of surface dS
– an outward unit normal vector $\mathbf{n}$

Traction $\mathbf{t}$ for the normal vector $\mathbf{n}$ is defined as

$$\mathbf{t}(\mathbf{n}) = \lim_{dS \to 0} \frac{\mathbf{F}}{dS}.$$

Definition of Stress

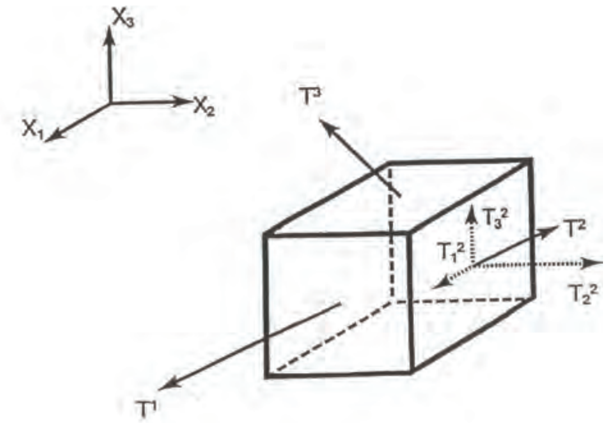


Figure 1.2 (from Geller, 1993)

$\mathbf{t}^{(j)}$ is traction for a normal vector in the x_j direction

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} {t_1}^{(1)} & {t_1}^{(2)} & {t_1}^{(3)} \\ {t_2}^{(1)} & {t_2}^{(2)} & {t_2}^{(3)} \\ {t_3}^{(1)} & {t_3}^{(2)} & {t_3}^{(3)} \end{pmatrix}$$

A general way of describing the forces acting within the body

Sign of Stress Components

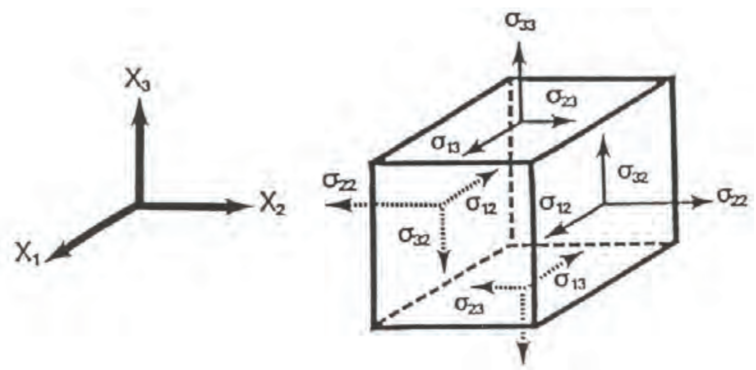


Figure 1.3 (from Geller, 1993)

– a positive stress is a force pointing in the direction associated with the outward normal.
– σ_{11}, σ_{22}, σ_{33}: normal tractions
  (σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31}, σ_{32}: shear tractions)
– positive normal traction = tension
  negative normal traction = compression

Traction on an Arbitrary Surface

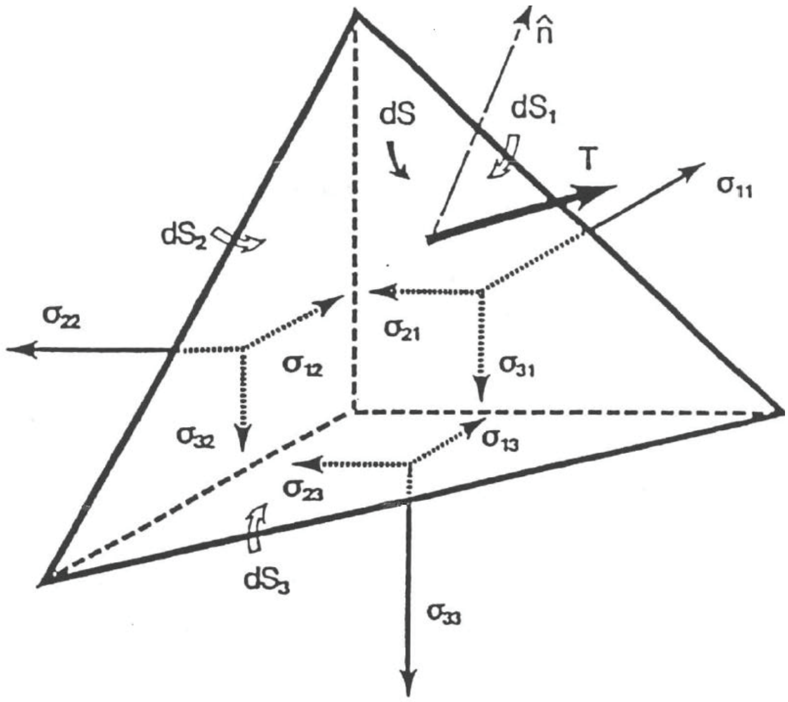


Figure 1.4 (from Geller, 1993)

Consider the infinitesimal tetrahedron with an arbitrarily oriented surface (area: dS, normal vector: $\mathbf{n}$).

$$\boxed{t_i = \sum_{j=1}^{3} \sigma_{ij} n_j}$$

Nature of Stress

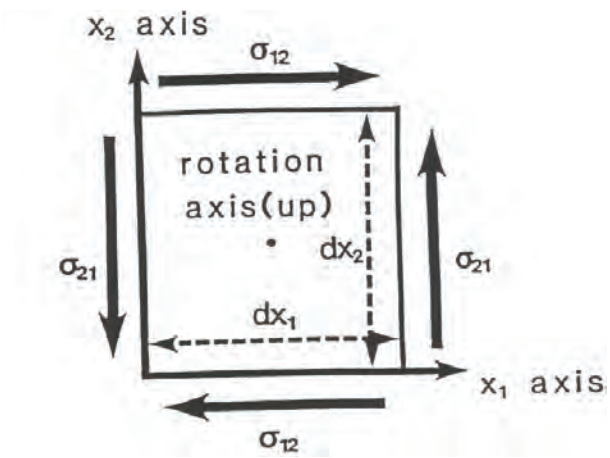


Figure 1.5 (from Geller, 1993)

$\boxed{\sigma_{ij} = \sigma_{ji}}$ (net torque should be zero)

1.2 Strain

Definition of Strain

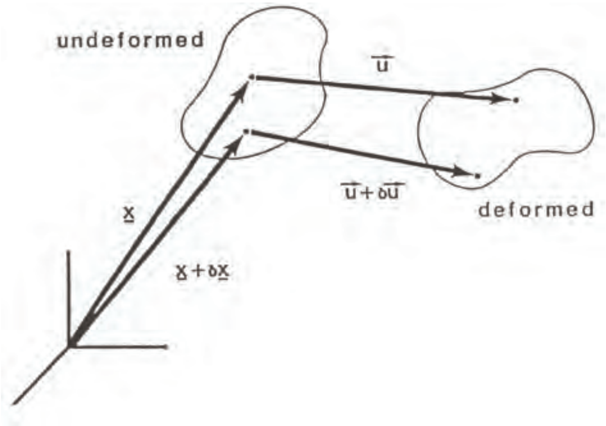


Figure 1.6 (from Geller, 1993)

<differential motion of adjacent particles>

$$\begin{aligned} \delta u_i(\mathbf{x}) &= u_i(\mathbf{x} + \boldsymbol{\delta}\mathbf{x}) - u_i(\mathbf{x}) \\ &= \sum_{j=1}^{3} u_{i,j}(\mathbf{x})\, \delta x_j \end{aligned}$$

We decompose as

$$\delta u_i = \sum_{j=1}^{3} \left(e_{ij} + w_{ij}\right)\, \delta x_j,$$

where

$$\begin{aligned} e_{ij} &= \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \text{and} \\ w_{ij} &= \frac{1}{2}\left(u_{i,j} - u_{j,i}\right). \end{aligned}$$

w_{ij}: components of rigid rotations
e_{ij}: components of deformations (**strain tensor**)

$$e_{ij} = \begin{pmatrix} u_{1,1} & \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) & \frac{1}{2}\left(u_{1,3} + u_{3,1}\right) \\ \frac{1}{2}\left(u_{2,1} + u_{1,2}\right) & u_{2,2} & \frac{1}{2}\left(u_{2,3} + u_{3,2}\right) \\ \frac{1}{2}\left(u_{3,1} + u_{1,3}\right) & \frac{1}{2}\left(u_{3,2} + u_{2,3}\right) & u_{3,3} \end{pmatrix}$$

$$(\text{dilatation: } \theta = \sum_{i=1}^{3} e_{i,i} = \sum_{i=1}^{3} u_{i,i} = \Delta V / V)$$

Note

$$e_{ij} = e_{ji}.$$

Various Displacement Fields

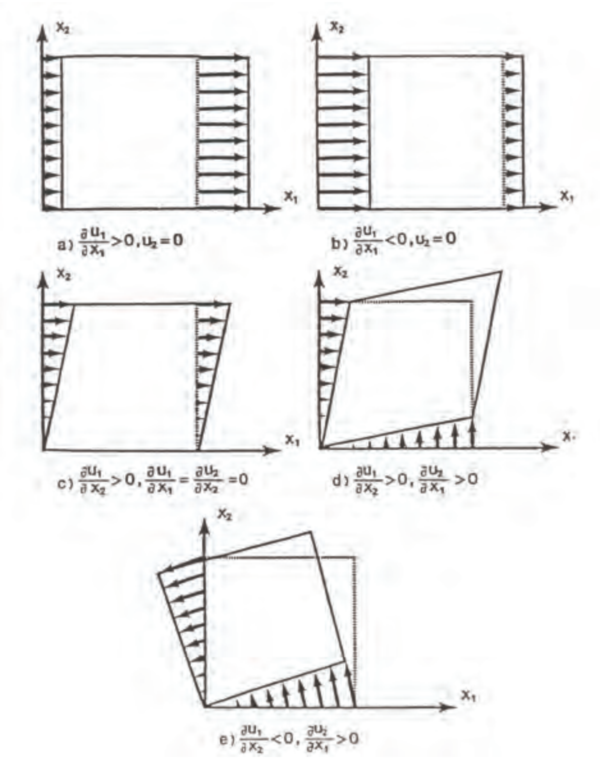


Figure 1.7 (from Geller, 1993)

$e_{11} \neq 0, \quad w_{ij} = 0$

$e_{12} \neq 0, \quad w_{12} \neq 0$ (left)
$e_{12} \neq 0, \quad w_{ij} = 0$ (right)

$e_{ij} = 0, \quad w_{12} \neq 0$

1.3 Basic Equations

1.3.1 Constitutive Equations

Hooke's law (isotropic media)

$$\boxed{\sigma_{ij} = \delta_{ij} \sum_{k=1}^{3} \lambda\, e_{kk} + 2\mu\, e_{ij}}$$

2-D case

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{pmatrix}$$

3-D case

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{pmatrix}$$

Hooke's law (general media)

$$\boxed{\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\, e_{kl}}$$

C_{ijkl}: elastic moduli
(4-th order tensor, 81 components)

1-D string case

$$\sigma = C\ e\,(= C\ u_{,x})$$

2-D case

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1112} & C_{1121} \\ C_{2211} & C_{2222} & C_{2212} & C_{2221} \\ C_{1211} & C_{1222} & C_{1212} & C_{1221} \\ C_{2111} & C_{2122} & C_{2112} & C_{2121} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{12} \\ e_{21} \end{pmatrix}$$

3-D case

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} & C_{3332} & C_{3313} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} & C_{2332} & C_{2313} & C_{2321} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} & C_{3132} & C_{3113} & C_{3121} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} & C_{1232} & C_{1213} & C_{1221} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3231} & C_{3212} & C_{3232} & C_{3213} & C_{3221} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1331} & C_{1312} & C_{1332} & C_{1313} & C_{1321} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \\ e_{32} \\ e_{13} \\ e_{21} \end{pmatrix}$$

Symmetry of Elastic Moduli

Because $\sigma_{ij} = \sigma_{ji}$ and $e_{kl} = e_{lk}$,

$$C_{ijkl} = C_{jikl} \quad \text{and} \quad C_{ijkl} = C_{ijlk}.$$

81 $\rightarrow$ 36 independent components

We usually further assume

$$C_{ijkl} = C_{klij}.$$

36 $\rightarrow$ 21 independent components

2-D case

$$\begin{pmatrix} C_{1111} & C_{1122} & C_{1112} \\ & C_{2222} & C_{2212} \\ & & C_{1212} \end{pmatrix} \text{ are independent components.}$$

3-D case

$$\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ & & & C_{2323} & C_{2331} & C_{2312} \\ & & & & C_{3131} & C_{3112} \\ & & & & & C_{1212} \end{pmatrix} \text{ are independent components.}$$

1.3.2 Dynamic Equation of Motion

Consider a block of material bounded by surfaces parallel to the coordinate axes.

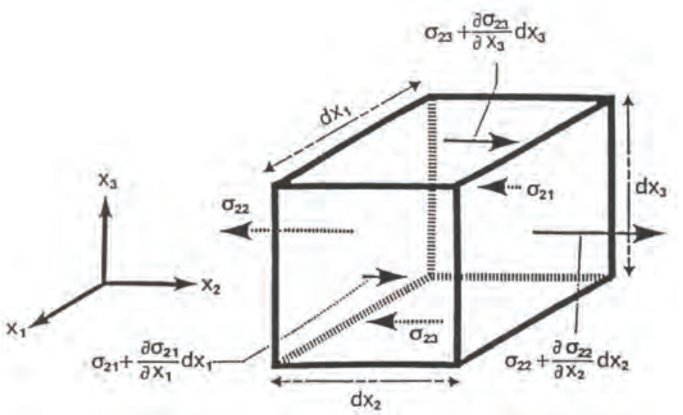


Figure 1.8 (from Geller, 1993)

<inertial force in the x_2 direction>

$$\rho\,\ddot{u}_2\,dx_1\,dx_2\,dx_3$$

<external force in the x_2 direction>

$$f_2\,dx_1\,dx_2\,dx_3$$

<stresses in the x_2 direction>

$$\begin{aligned}
&\sigma_{21}\left(x_1+dx_1/2,x_2,x_3\right)dx_2dx_3-\sigma_{21}\left(x_1-dx_1/2,x_2,x_3\right)dx_2dx_3\\
&+\sigma_{22}\left(x_1,x_2+dx_2/2,x_3\right)dx_1dx_3-\sigma_{22}\left(x_1,x_2-dx_2/2,x_3\right)dx_1dx_3\\
&+\sigma_{23}\left(x_1,x_2,x_3+dx_3/2\right)dx_1dx_2-\sigma_{23}\left(x_1,x_2,x_3-dx_3/2\right)dx_1dx_2\\
&\quad=\ \sigma_{21,1}\,dx_1dx_2dx_3+\sigma_{22,2}\,dx_1dx_2dx_3+\sigma_{23,3}\,dx_1dx_2dx_3\\
&\quad=\ \sum_{j=1}^{3}\sigma_{2j,j}\,dx_1dx_2dx_3
\end{aligned}$$

Thus, the dynamic equation of motion can be written as

$$\begin{aligned}
\rho\,\ddot{u}_1\,dx_1dx_2dx_3 &= \sum_{j=1}^{3}\sigma_{1j,j}\,dx_1dx_2dx_3+f_1\,dx_1dx_2dx_3\\
\rho\,\ddot{u}_2\,dx_1dx_2dx_3 &= \sum_{j=1}^{3}\sigma_{2j,j}\,dx_1dx_2dx_3+f_2\,dx_1dx_2dx_3\\
\rho\,\ddot{u}_3\,dx_1dx_2dx_3 &= \sum_{j=1}^{3}\sigma_{3j,j}\,dx_1dx_2dx_3+f_3\,dx_1dx_2dx_3
\end{aligned}$$

or

$$\boxed{\rho\,\ddot{u}_i=\sum_{j=1}^{3}\sigma_{ij,j}+f_i}$$

(3 equations for 9 unknowns)

1.3.3 Summary

$$
\begin{aligned}
\rho\,\ddot{u}_i &= \sum_{j=1}^{3} \sigma_{ij,j} + f_i \\
\sigma_{ij} &= \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\,e_{kl} \\
e_{kl} &= \frac{1}{2}\left(u_{k,l} + u_{l,k}\right)
\end{aligned}
$$

or

$$
\boxed{\rho\,\ddot{u}_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \left(C_{ijkl}u_{k,l}\right)_{,j} + f_i}
$$

(3 equations for 3 unknowns)

1.4 Boundary Conditions

<surface>

free surface conditions:

$t_i = 0$

for horizontal surfaces:

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

<solid-solid interfaces>

continuity of traction and displacement:

${t_i}^+ = {t_i}^-, \quad {u_i}^+ = {u_i}^-$

for horizontal interfaces:

${\sigma_{i3}}^+ = {\sigma_{i3}}^-, \quad {u_i}^+ = {u_i}^-$

<solid-liquid interfaces>

$$\sum_{j=1}^{3} {t_j}^+ n_j = \sum_{j=1}^{3} {t_j}^- n_j, \quad \sum_{j=1}^{3} {u_j}^+ n_j = \sum_{j=1}^{3} {u_j}^- n_j$$

$${t_i}^+ - n_i \sum_{j=1}^{3} {t_j}^+ n_j = 0$$

($^+$ denote the solid side, n_i points to the liquid)

for horizontal interfaces:

${t_3}^+ = {t_3}^-, \quad {u_3}^+ = {u_3}^-,$

${t_1}^+ = {t_2}^+ = 0$

1.5 Useful Variables and Concepts

1.5.1 Stress Components in Different Coordinate System

e.g.)

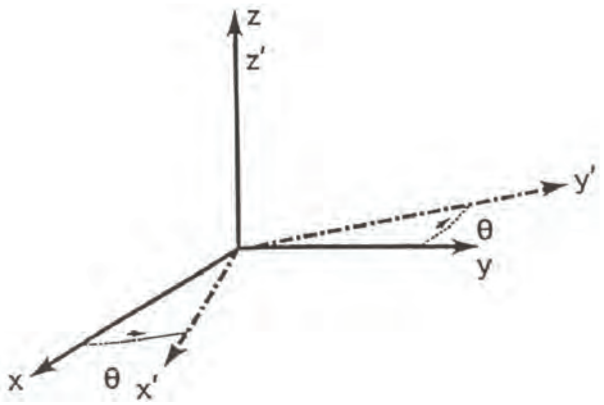


Figure 1.9 (from Geller, 1993)

$$\begin{aligned}\mathbf{a} &= x_1\mathbf{e}_x + x_2\mathbf{e}_y + x_3\mathbf{e}_z \\ &= {x_1}'\mathbf{e}_{x'} + {x_2}'\mathbf{e}_{y'} + {x_3}'\mathbf{e}_{z'}\end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \end{pmatrix}$$

$$\mathbf{x} = \mathbf{A}\,\mathbf{x}'$$

traction and normal vectors in the new and old coordinate system:

$$\mathbf{t} = \mathbf{A}\,\mathbf{t}', \quad \mathbf{n} = \mathbf{A}\,\mathbf{n}'$$

relations between stress tensors and traction:

$$\mathbf{t} = \sigma\,\mathbf{n}, \quad \mathbf{t}' = \sigma'\,\mathbf{n}'$$

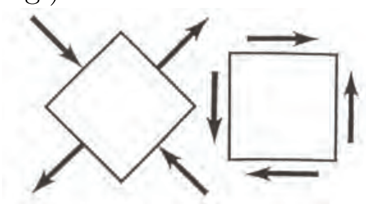
Thus we have

$$\boxed{\sigma' = A^{-1}\,\sigma\,A}$$

For rotational coordinate transformations,

$$\boxed{\sigma' = A^{T}\,\sigma\,A}$$

e.g.)



$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Figure 1.10
(from Geller, 1993)

$$\begin{pmatrix} {x_1}' \\ {x_2}' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Thus

$$\begin{aligned}\sigma' &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

1.5.2 Principal Stresses

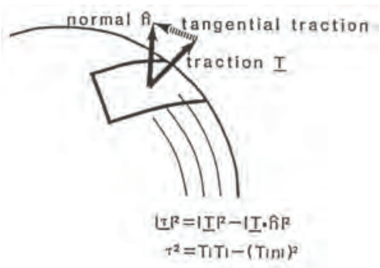


Figure 1.11 (from Geller, 1993)

<tangential traction>
components along the surface

<normal traction>
component perpendicular to the surface

<principal stress axes>
normal vectors of the surfaces where tangential tractions are zero.

<u>How to Find the Principal Axes and Stresses</u>
Principal axes **n** and principal stresses λ satisfies

$$(t_i =) \sum_{j=1}^{3} \sigma_{ij}\, n_j = \lambda\, n_i.$$

Thus, principal axes and stresses are eigenvectors and eigenvalues, respectively, of stress tensors σ, which are non-trivial solutions for the following equation

$$\sum_{j=1}^{3} (\sigma_{ij} - \lambda\, \delta_{ij})\; n_j = 0.$$

e.g.) principal axes and stresses for $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$|\sigma - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad \lambda = 1, -1$$

$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ are eigenvectors

for $\lambda = 1$ and $\lambda = -1$, respectively.

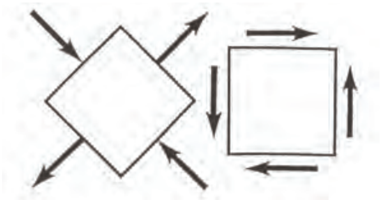


Figure 1.12 (from Geller, 1993)

e.g.) $\sigma = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad \lambda = 1, 3, 4$$

$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{0}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$, and $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ are eigenvectors.

<u>Stress and Traction in the Principal Axes</u>

Consider we use coordinate system of the principal axes.
Stress tensors can be written as

$$\sigma = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}$$

Traction on an arbitrary surface of the normal vector $\mathbf{n}$ can be therefore written as

$$t_i = \lambda^{(i)} n_i.$$

1.5.3 Surface with Maximum Tangential Traction

Consider we use coordinate system of the principal axes.
Tangential traction τ on a surface of the normal vector $\mathbf{n}$ can be computed as

$$\begin{aligned} \tau^2 &= \sum_i t_i t_i - \left(\sum_i t_i n_i \right)^2 \\ &= \sum_i \left(\lambda^{(i)} n_i \right)^2 - \left(\sum_i \lambda^{(i)} {n_i}^2 \right)^2. \end{aligned}$$

$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ are $\mathbf{n}$s to give local maxima of τ.

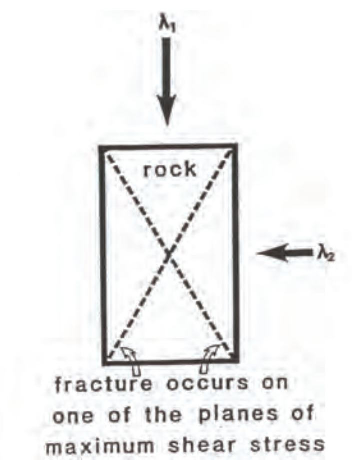


Figure 1.13 (from Geller, 1993)

If $\lambda^{(1)} < \lambda^{(2)} < \lambda^{(3)}$, global maxima is

$$\tau = \frac{\lambda^{(3)} - \lambda^{(1)}}{2}$$

on the surface of

$$\mathbf{n} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Insight to Faulting

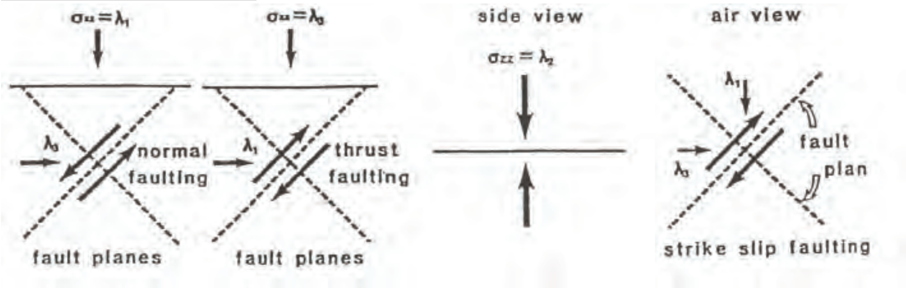


Figure 1.14 (from Geller, 1993)

$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	
v	h	h	normal faulting
h	h	v	thrust faulting
h	v	h	strike slip

v: vertical
h: horizontal

1.5.4 Deviatoric Stresses

<mean stress>

$$P = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

<deviatoric stress>

$$D_{ij} = \sigma_{ij} - P\,\delta_{ij},$$

which are deviations of the stress from the mean (compressional) stress.

1.5.5 Kinetic and Strain Energy

<kinetic energy T>

$$T = \frac{1}{2}\int \sum_i \dot{u}_i\,\rho\,\dot{u}_i\,dV$$

<strain energy W>

$$W = \frac{1}{2}\int \sum_i \sum_j \sum_k \sum_l e_{ij}\,C_{ijkl}\,e_{kl}\,dV$$

or

$$W = \frac{1}{2}\int \sum_i \sum_j \sum_k \sum_l u_{i,j}\,C_{ijkl}\,u_{k,l}\,dV$$

Chapter 2

Elastic Waves

2.1 Derivation of Elastic Wave Equation

Elastc Equation of Motion (without a source term):

$$\sum_j \sum_k \sum_l \left(C_{ijkl}\, u_{k,l}\right)_{,j} - \rho\, \frac{\partial^2 u_i}{\partial t^2} = 0$$

For homogeneous isortropic media, we have

$$(\lambda + \mu)\ \nabla\ (\nabla \cdot \mathbf{u}) + \mu\, \nabla^2 \mathbf{u} - \rho\, \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0.$$

Using $\nabla^2 \mathbf{u} = \nabla\ (\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$, we have

$$(\lambda + 2\mu)\ \nabla\ (\nabla \cdot \mathbf{u}) - \mu\, \nabla \times \nabla \times \mathbf{u} - \rho\, \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0.$$

We represent the vector field $\mathbf{u}$ by using a scalar potential γ and a vector potential $\boldsymbol{\Psi}$:

$$\mathbf{u} = \nabla \gamma + \nabla \times \boldsymbol{\Psi}.$$

Substituing the above representation, we have

$$(\lambda + 2\mu)\ \nabla\ \left(\nabla^2 \gamma\right) - \mu\, \nabla \times \nabla \times \nabla \times \boldsymbol{\Psi} - \rho\, \frac{\partial^2}{\partial t^2}\, (\nabla \gamma + \nabla \times \boldsymbol{\Psi}) = 0.$$

Rearranging terms, we have

$$\nabla \left[(\lambda + 2\mu)\ \nabla^2 \gamma - \rho\, \frac{\partial^2 \gamma}{\partial t^2} \right] + \nabla \times \left[\mu\, \nabla^2 \boldsymbol{\Psi} - \rho\, \frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} \right] = 0.$$

We thus have

$$(\lambda + 2\mu)\ \nabla^2 \gamma - \rho\, \frac{\partial^2 \gamma}{\partial t^2} = 0, \qquad \mu\, \nabla^2 \boldsymbol{\Psi} - \rho\, \frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} = 0,$$

or

$$\nabla^2 \gamma - \frac{1}{\alpha^2}\, \frac{\partial^2 \gamma}{\partial t^2} = 0, \qquad \nabla^2 \boldsymbol{\Psi} - \frac{1}{\beta^2}\, \frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} = 0,$$

where

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad \beta = \sqrt{\frac{\mu}{\rho}}.$$

2.2 The Wave Equation

<one dimensional scalar wave equation> (in a homogeneous medium)

$$\frac{\partial^2 \gamma\,(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \gamma}{\partial t^2}\,(x,t) = 0$$

General solutions:

$$\gamma(x,t) = A\, f\,(t + x/v) + B\, g\,(t - x/v)$$

e.g.)

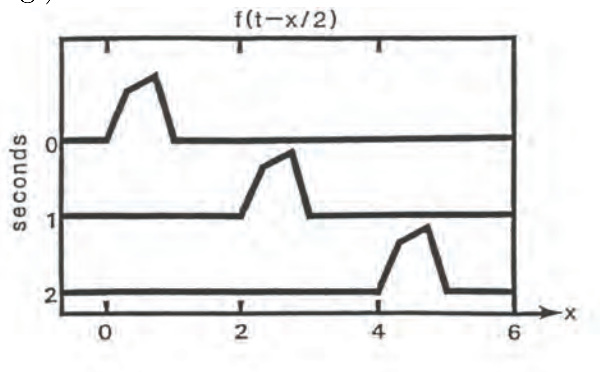


Figure 2.1 (from Geller, 1993)

<useful form of solutions>
a monochromatic (single frequency) wave

$$\begin{aligned} \gamma &= A\,\exp\left(i\omega\,(t \pm x/v)\right) \\ &= A\,\exp\left(i\,(\omega t \pm kx)\right) \\ &= A\,\exp\left(2\pi i\left(\frac{t}{T} \pm \frac{x}{\lambda}\right)\right) \end{aligned}$$

<Commonly Used Wave Variables>

	units	
Velocity	m/s	$v = \omega/k = f\lambda$
Frequency	1/s	$f = \omega/(2\pi) = 1/T = v/\lambda$
Angular Frequency	1/s	$\omega = 2\pi f = 2\pi/T$
Period	s	$T = 2\pi/\omega = 1/f$
Wavelength	m	$\lambda = 2\pi/k = v/f = vT$
Wavenumber	1/m	$k = 2\pi/\lambda = \omega/v$

<3-D scalar wave equation> (in a homogeneous medium)

$$\nabla^2 \gamma\,(\mathbf{x},t) - \frac{1}{\alpha^2}\frac{\partial^2 \gamma}{\partial t^2}\,(\mathbf{x},t) = 0$$

a monochromatic (single frequency) plane wave

$$\gamma\,(\mathbf{x},t) \;=\; A\,\exp\left(i\,(\omega t \pm \mathbf{k}\cdot\mathbf{x})\right), \quad |\mathbf{k}| = \omega/\alpha$$

<3-D vector wave equation> (in a homogeneous medium)

$$\nabla^2 \mathbf{\Psi}(\mathbf{x}, t) - \frac{1}{\beta^2}\frac{\partial^2 \mathbf{\Psi}}{\partial t^2}(\mathbf{x}, t) = 0$$

a monochromatic (single frequency) plane wave

$$\mathbf{\Psi}(\mathbf{x}, t) \quad = \quad \mathbf{A}\ \exp\left(i\left(\omega t \pm \mathbf{k}\cdot\mathbf{x}\right)\right), \qquad |\mathbf{k}| = \omega/\beta$$

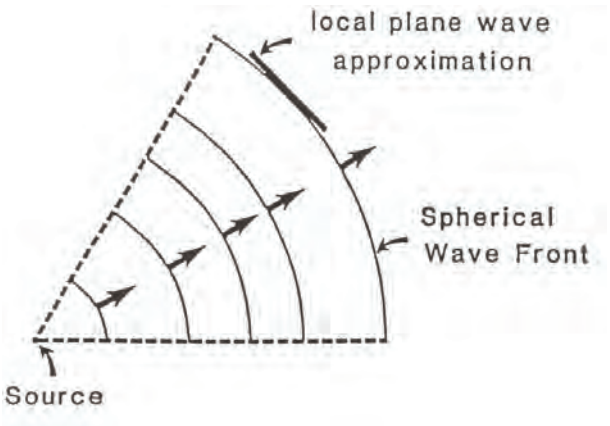


Figure 2.2 (from Geller, 1993)

2.3 P and S Waves

<P wave>

$$\nabla^2\gamma - \frac{1}{\alpha^2}\frac{\partial^2\gamma}{\partial t^2} = 0, \qquad \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

a plane wave traveling in the $+z$ direction

$$\gamma(z, t) = A\ \exp\left(i\left(\omega t - kz\right)\right), \quad k = \omega/\alpha$$

$$\mathbf{u} = \nabla\gamma = (0, 0, -ikA)\exp\left(i\left(\omega t - kz\right)\right)$$

The displacement is along the propagation direction.

<S wave>

$$\nabla^2\mathbf{\Psi} - \frac{1}{\beta^2}\frac{\partial^2\mathbf{\Psi}}{\partial t^2} = 0, \qquad \beta = \sqrt{\frac{\mu}{\rho}}.$$

a plane wave traveling in the $+z$ direction

$$\mathbf{\Psi}(z, t) = \mathbf{A}\ \exp\left(i\left(\omega t - kz\right)\right), \quad k = \omega/\beta$$

$$\mathbf{u} = \nabla\times\mathbf{\Psi} = (ikA_y, -ikA_x, 0)\exp\left(i\left(\omega t - kz\right)\right)$$

The displacement along the propagation direction is zero.

$$\theta = \nabla\cdot\mathbf{u} = 0$$

S wave causes no volume change.

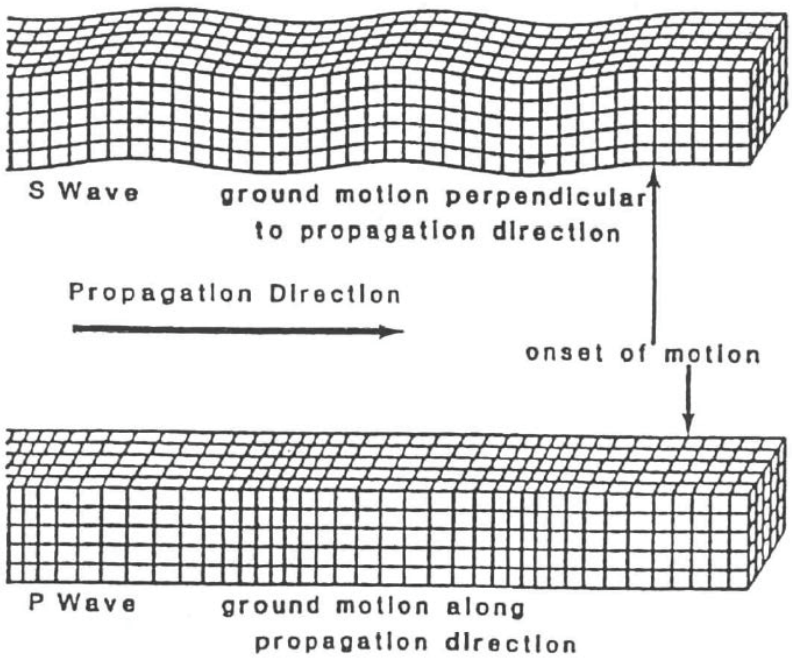


Figure 2.3 (from Geller, 1993)

<SV and SH waves>

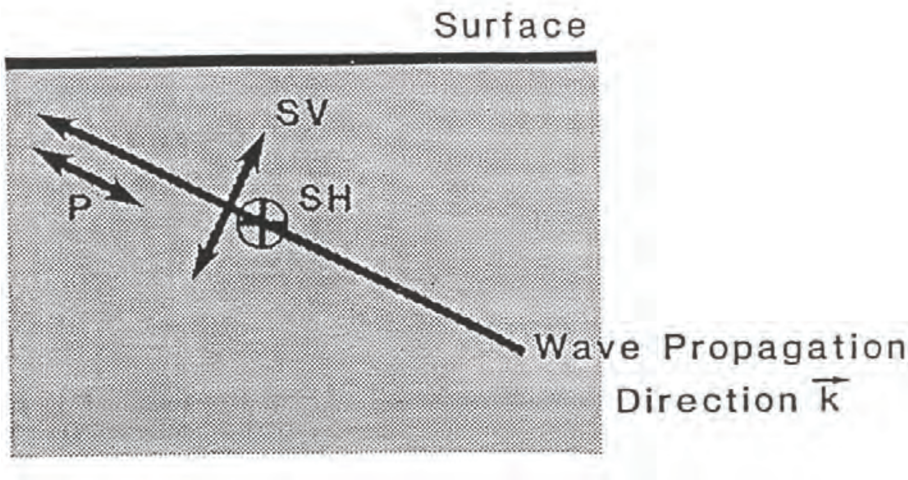


Figure 2.4 (from Geller, 1993)

SV: component along a vertical plane that is parallel to the wave number vector
SH: component parpendicular to the vertical plane

Chapter 3

Reflection and Refraction

3.1 SNELL'S LAW

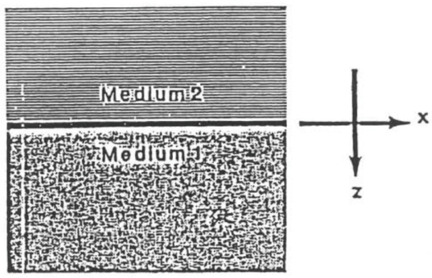


Figure 3.1 (from Geller, 1993)

Apparent Velocity

$$c_x = \frac{v}{\sin i} = \frac{\omega}{k_x} \qquad (i\text{: incident angle})$$

Snell's Law

$$c_x = \frac{\alpha_1}{\sin i_1} = \frac{\beta_1}{\sin j_1} = \frac{\alpha_2}{\sin i_2} = \frac{\beta_2}{\sin j_2}$$

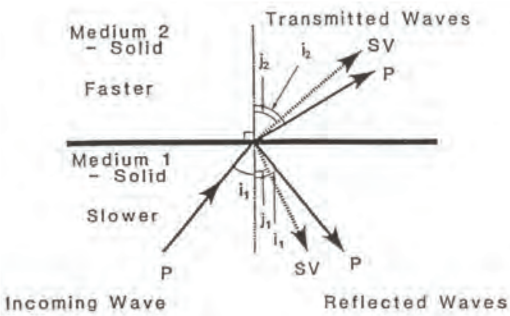


Figure 3.2 (from Geller, 1993)

Ray Parameter

$$p = \frac{1}{c_x} = \frac{\sin i}{v}$$

- is often more useful than the apparent velocity
- p is always constant for any ray in a laterally homogeneous medium

Critical Angle

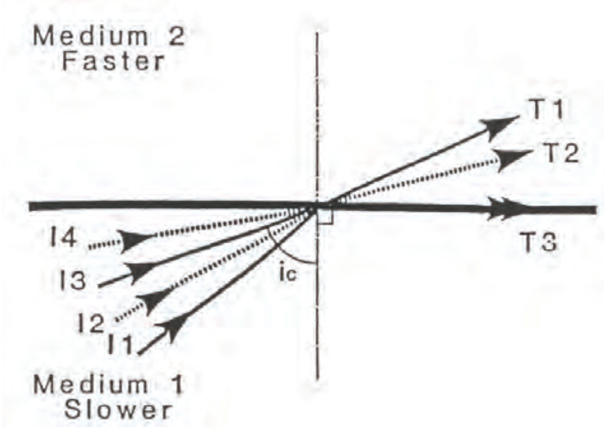


Figure 3.3 (from Geller, 1993)

$$i_2 = \sin^{-1}\left[\frac{\alpha_2}{\alpha_1}(\sin i_1)\right] \text{ when } \frac{\alpha_2}{\alpha_1}(\sin i_1) < 1.$$

$$\sin i_c = \frac{\alpha_1}{\alpha_2} \quad (i_c : \text{critical angle})\,.$$

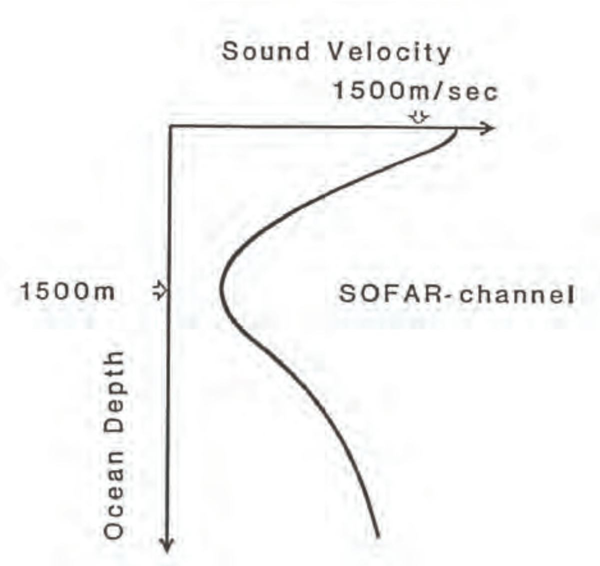


Figure 3.4 (from Geller, 1993)

3.2 Reflection and Transmission Coefficients

SH Waves

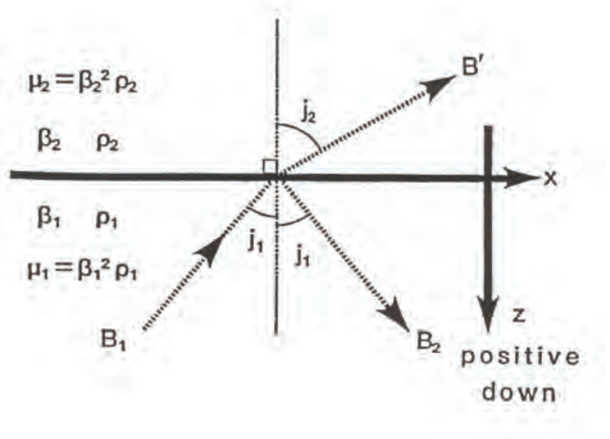


Figure 3.5 (from Geller, 1993)

$$\begin{aligned} u_y^1 &= \acute{B}_1 \exp\left(i\left(\omega t - k_x x + k_z^1 z\right)\right) + \grave{B}_1 \exp\left(i\left(\omega t - k_x x - k_z^1 z\right)\right) \\ u_y^2 &= \acute{B}_2 \exp\left(i\left(\omega t - k_x x + k_z^2 z\right)\right) \end{aligned}$$

$$\rho_1\, \omega^2 = \mu_1 \left({k_x}^2 + {k_z^1}^2\right), \qquad \rho_2\, \omega^2 = \mu_2 \left({k_x}^2 + {k_z^2}^2\right)$$

Boundary conditions at $z = 0$:
<displacement continuity>

$$\acute{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) + \grave{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) = \acute{B}_2 \exp\left(i\left(\omega t - k_x x\right)\right)$$

<traction continuity>

$$\begin{aligned} & ik_z^1 \mu_1 \acute{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) - ik_z^1 \mu_1 \grave{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) \\ = \;& ik_z^2 \mu_2 \acute{B}_2 \exp\left(i\left(\omega t - k_x x\right)\right) \end{aligned}$$

Transmission Coefficient

$$T = \frac{\acute{B}_2}{\acute{B}_1} = \frac{2\mu_1 k_z^1}{\mu_1 k_z^1 + \mu_2 k_z^2}$$

Reflection Coefficient

$$R = \frac{\grave{B}_1}{\acute{B}_1} = \frac{\mu_1 k_z^1 - \mu_2 k_z^2}{\mu_1 k_z^1 + \mu_2 k_z^2}$$

or

$$T = \frac{2\rho_1 \beta_1 \cos j_1}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2}$$

$$R = \frac{\rho_1 \beta_1 \cos j_1 - \rho_2 \beta_2 \cos j_2}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2}$$

SH waves at vertical incidence

$$T = \frac{2\rho_1 \beta_1}{\rho_1 \beta_1 + \rho_2 \beta_2}$$

$$R = \frac{\rho_1 \beta_1 - \rho_2 \beta_2}{\rho_1 \beta_1 + \rho_2 \beta_2}$$

$$\rho\beta\text{: acoustic impedence}$$

Free Surface

$\mu_2 = \beta_2 = 0$

$$R = 1$$

Post-Critical Waves

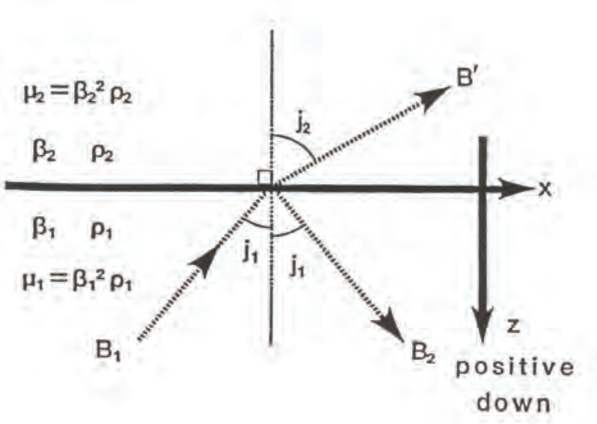


Figure 3.6 (from Geller, 1993)

$$
\begin{aligned}
u_y^1 &= \acute{B}_1 \exp\left(i\left(\omega t - k_x x + k_z^1 z\right)\right) + \grave{B}_1 \exp\left(i\left(\omega t - k_x x - k_z^1 z\right)\right) \\
u_y^2 &= \acute{B}_2 \exp\left(i\left(\omega t - k_x x\right) + k_z^{2*} z\right)
\end{aligned}
$$

$$\rho_1\, \omega^2 = \mu_1 \left({k_x}^2 + {k_z^1}^2\right), \qquad \rho_2\, \omega^2 = \mu_2 \left({k_x}^2 - {k_z^{2*}}^2\right)$$

Boundary conditions at $z = 0$:

<displacement continuity>

$$\acute{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) + \grave{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) = \acute{B}_2 \exp\left(i\left(\omega t - k_x x\right)\right)$$

<traction continuity>

$$
\begin{aligned}
& i k_z^1 \mu_1 \acute{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) - i k_z^1 \mu_1 \grave{B}_1 \exp\left(i\left(\omega t - k_x x\right)\right) \\
= \; & k_z^{2*} \mu_2 \acute{B}_2 \exp\left(i\left(\omega t - k_x x\right)\right)
\end{aligned}
$$

Transmission Coefficient

$$T = \frac{\acute{B}_2}{\acute{B}_1} = \frac{2\mu_1 k_z^1}{\mu_1 k_z^1 - i\mu_2 k_z^{2*}}$$

Reflection Coefficient

$$R = \frac{\grave{B}_1}{\acute{B}_1} = \frac{\mu_1 k_z^1 + i\mu_2 k_z^{2*}}{\mu_1 k_z^1 - i\mu_2 k_z^{2*}} \qquad (|R| = 1)$$

- The amplitude of the transmitted wave decays as the distance from the interface inreases.
- no vertical energy transportation in the upper layer
- phase shift in the reflected wave

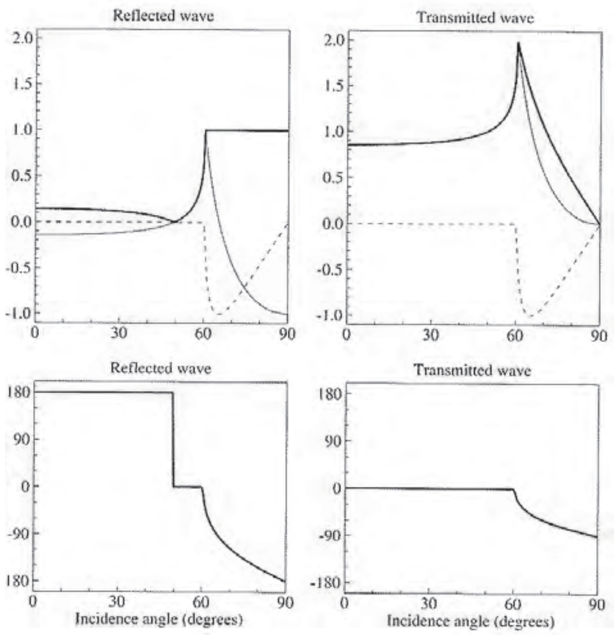


Figure 3.7 (from Shearer, P.M., 1999)

P-SV Waves

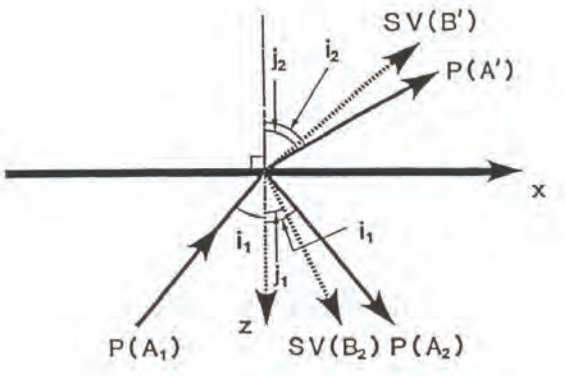


Figure 3.8 (from Geller, 1993)

$$\mathbf{u} = \nabla\gamma + \nabla \times \begin{pmatrix} 0 \\ \Psi \\ 0 \end{pmatrix}$$

$$\begin{aligned} \gamma^1 &= \acute{A}_1 \exp\left(i\left(\omega t - k_x x + k_{z\alpha}^1 z\right)\right) + \grave{A}_1 \exp\left(i\left(\omega t - k_x x - k_{z\alpha}^1 z\right)\right) \\ \gamma^2 &= \acute{A}_2 \exp\left(i\left(\omega t - k_x x + k_{z\alpha}^2 z\right)\right) \end{aligned}$$

$$\begin{aligned}\Psi^1 &= \grave{B}_1 \exp\left(i\left(\omega t - k_x x - k^1_{z\beta} z\right)\right) \\ \Psi^2 &= \acute{B}_2 \exp\left(i\left(\omega t - k_x x + k^2_{z\beta} z\right)\right)\end{aligned}$$

$$\begin{aligned}\rho_1\,\omega^2 &= (\lambda_1 + 2\mu_1)\left({k_x}^2 + {k^1_{z\alpha}}^2\right) \\ \rho_2\,\omega^2 &= (\lambda_2 + 2\mu_2)\left({k_x}^2 + {k^2_{z\alpha}}^2\right) \\ \rho_1\,\omega^2 &= \mu_1\left({k_x}^2 + {k^1_{z\beta}}^2\right) \\ \rho_2\,\omega^2 &= \mu_2\left({k_x}^2 + {k^2_{z\beta}}^2\right)\end{aligned}$$

$$u^1_x = \frac{\partial\gamma^1}{\partial x} - \frac{\partial\Psi^1}{\partial z}, \quad u^1_z = \frac{\partial\gamma^1}{\partial z} + \frac{\partial\Psi^1}{\partial x}$$

$$u^2_x = \frac{\partial\gamma^2}{\partial x} - \frac{\partial\Psi^2}{\partial z}, \quad u^2_z = \frac{\partial\gamma^2}{\partial z} + \frac{\partial\Psi^2}{\partial x}$$

$$\begin{aligned}\sigma^1_{xz} &= \mu_1\left(2\frac{\partial^2\gamma^1}{\partial x\partial z} - \frac{\partial^2\Psi^1}{\partial z^2} + \frac{\partial^2\Psi^1}{\partial x^2}\right) \\ \sigma^1_{zz} &= \lambda_1\nabla^2\gamma^1 + 2\mu_1\left(\frac{\partial^2\gamma^1}{\partial z^2} + \frac{\partial^2\Psi^1}{\partial x\partial z}\right) \\ \sigma^2_{xz} &= \mu_2\left(2\frac{\partial^2\gamma^2}{\partial x\partial z} - \frac{\partial^2\Psi^2}{\partial z^2} + \frac{\partial^2\Psi^2}{\partial x^2}\right) \\ \sigma^2_{zz} &= \lambda_2\nabla^2\gamma^2 + 2\mu_2\left(\frac{\partial^2\gamma^2}{\partial z^2} + \frac{\partial^2\Psi^2}{\partial x\partial z}\right)\end{aligned}$$

Bounday conditions at $z = 0$:

<displaycement>

$$\begin{aligned}-ik_x\acute{A}_1 - ik_x\grave{A}_1 + ik^1_{z\beta}\grave{B}_1 &= -ik_x\acute{A}_2 - ik^2_{z\beta}\acute{B}_2 \\ ik^1_{z\alpha}\acute{A}_1 - ik^2_{z\alpha}\grave{A}_1 - ik_x\grave{B}_1 &= ik^2_{z\alpha}\acute{A}_2 - ik_x\acute{B}_2\end{aligned}$$

<traction>

$$\begin{aligned}&\mu_1\left\{2(-ik_x)\left(ik^1_{z\alpha}\right)\acute{A}_1 + 2(-ik_x)\left(-ik^1_{z\alpha}\right)\grave{A}_1\right. \\ &\qquad\left. - \left(-ik^1_{z\beta}\right)\left(-ik^1_{z\beta}\right)\grave{B}_1 + (-ik_x)(-ik_x)\grave{B}_1\right\} \\ &\quad = \mu_2\left\{2(-ik_x)\left(ik^2_{z\alpha}\right)\acute{A}_2 - \left(ik^2_{z\beta}\right)\left(ik^2_{z\beta}\right)\acute{B}_2 + (-ik_x)(-ik_x)\acute{B}_2\right\} \\ &\lambda_1\left\{(-ik_x)(-ik_x)\acute{A}_1 + \left(ik^1_{z\alpha}\right)\left(ik^1_{z\alpha}\right)\acute{A}_1\right. \\ &\qquad\left. + (-ik_x)(-ik_x)\grave{A}_1 + \left(-ik^1_{z\alpha}\right)\left(-ik^1_{z\alpha}\right)\grave{A}_1\right\} \\ &+2\mu_1\left\{\left(ik^1_{z\alpha}\right)\left(ik^1_{z\alpha}\right)\acute{A}_1 + \left(-ik^1_{z\alpha}\right)\left(-ik^1_{z\alpha}\right)\grave{A}_1 + (-ik_x)\left(-ik^1_{z\beta}\right)\grave{B}_1\right\} \\ &\quad = \lambda_2\left\{(-ik_x)(-ik_x)\acute{A}_2 + \left(ik^2_{z\alpha}\right)\left(ik^2_{z\alpha}\right)\acute{A}_2\right\} \\ &\qquad +2\mu_2\left\{\left(ik^2_{z\alpha}\right)\left(ik^2_{z\alpha}\right)\acute{A}_2 + (-ik_x)\left(ik^2_{z\beta}\right)\acute{B}_2\right\}\end{aligned}$$

In detail, these formulas make repeated use of the variables

$$
\begin{aligned}
a &= \rho_2\left(1-2\beta_2^2p^2\right)-\rho_1\left(1-2\beta_1^2p^2\right), & b &= \rho_2\left(1-2\beta_2^2p^2\right)+2\rho_1\beta_1^2p^2,\\
c &= \rho_1\left(1-2\beta_1^2p^2\right)+2\rho_2\beta_2^2p^2, & d &= 2\left(\rho_2\beta_2^2-\rho_1\beta_1^2\right),
\end{aligned}
$$

and repeated use also of the cosine-dependent terms

$$
\begin{aligned}
E &= b\frac{\cos i_1}{\alpha_1}+c\frac{\cos i_2}{\alpha_2}, & F &= b\frac{\cos j_1}{\beta_1}+c\frac{\cos j_2}{\beta_2},\\
G &= a-d\frac{\cos i_1}{\alpha_1}\frac{\cos j_2}{\beta_2}, & H &= a-d\frac{\cos i_2}{\alpha_2}\frac{\cos j_1}{\beta_1},\\
D &= EF+GHp^2 \;=\; (\det \mathbf{M})\,/\,(\alpha_1\alpha_2\beta_1\beta_2)
\end{aligned}
$$

The main formulas are

$$
\begin{aligned}
\grave{P}\acute{P} &= \left[\left(b\frac{\cos i_1}{\alpha_1}-c\frac{\cos i_2}{\alpha_2}\right)F-\left(a+d\frac{\cos i_1}{\alpha_1}\frac{\cos j_2}{\beta_2}\right)Hp^2\right]/D,\\
\grave{P}\acute{S} &= -2\frac{\cos i_1}{\alpha_1}\left(ab+cd\frac{\cos i_2}{\alpha_2}\frac{\cos j_2}{\beta_2}\right)p\alpha_1/\left(\beta_1 D\right),\\
\grave{P}\grave{P} &= 2\rho_1\frac{\cos i_1}{\alpha_1}F\alpha_1/\left(\alpha_2 D\right),\\
\grave{P}\grave{S} &= 2\rho_1\frac{\cos i_1}{\alpha_1}Hp\alpha_1/\left(\beta_2 D\right),\\
\grave{S}\acute{P} &= -2\frac{\cos j_1}{\beta_1}\left(ab+cd\frac{\cos i_2}{\alpha_2}\frac{\cos j_2}{\beta_2}\right)p\beta_1/\left(\alpha_1 D\right),\\
\grave{S}\acute{S} &= -\left[\left(b\frac{\cos j_1}{\beta_1}-c\frac{\cos j_2}{\beta_2}\right)E-\left(a+d\frac{\cos i_2}{\alpha_2}\frac{\cos j_1}{\beta_1}\right)Gp^2\right]/D,\\
\grave{S}\grave{P} &= -2\rho_1\frac{\cos j_1}{\beta_1}Gp\beta_1/\left(\alpha_2 D\right),\\
\grave{S}\grave{S} &= 2\rho_1\frac{\cos j_1}{\beta_1}E\beta_1/\left(\beta_2 D\right),\\
\acute{P}\acute{P} &= 2\rho_2\frac{\cos i_2}{\alpha_2}F\alpha_2/\left(\alpha_1 D\right),\\
\acute{P}\acute{S} &= -2\rho_2\frac{\cos i_2}{\alpha_2}Gp\alpha_2/\left(\alpha_2 D\right),\\
\acute{P}\grave{P} &= -\left[\left(b\frac{\cos i_1}{\alpha_1}-c\frac{\cos i_2}{\alpha_2}\right)F+\left(a+d\frac{\cos i_2}{\alpha_2}\frac{\cos j_1}{\beta_1}\right)Gp^2\right]/D,\\
\acute{P}\grave{S} &= 2\frac{\cos i_2}{\alpha_2}\left(ac+bd\frac{\cos i_1}{\alpha_1}\frac{\cos j_1}{\beta_1}\right)p\alpha_2/\left(\beta_2 D\right),\\
\acute{S}\acute{P} &= 2\rho_2\frac{\cos j_2}{\beta_2}Hp\beta_2/\left(\alpha_1 D\right),\\
\acute{S}\acute{S} &= 2\rho_2\frac{\cos j_2}{\beta_2}E\beta_2/\left(\beta_1 D\right),
\end{aligned}
$$

$$
\begin{aligned}
\acute{S}\grave{P} &= 2\frac{\cos j_2}{\beta_2}\left(ac + bd\frac{\cos i_1}{\alpha_1}\frac{\cos j_1}{\beta_1}\right)p\beta_2/\left(\alpha_2 D\right), \\
\acute{S}\grave{S} &= \left[\left(b\frac{\cos j_1}{\beta_1} - c\frac{\cos j_2}{\beta_2}\right)E + \left(a + d\frac{\cos i_1}{\alpha_1}\frac{\cos j_2}{\beta_2}\right)Hp^2\right]/D
\end{aligned}
$$

"Quantitative Seismology" by Aki & Richards

Chapter 4

Surface Waves

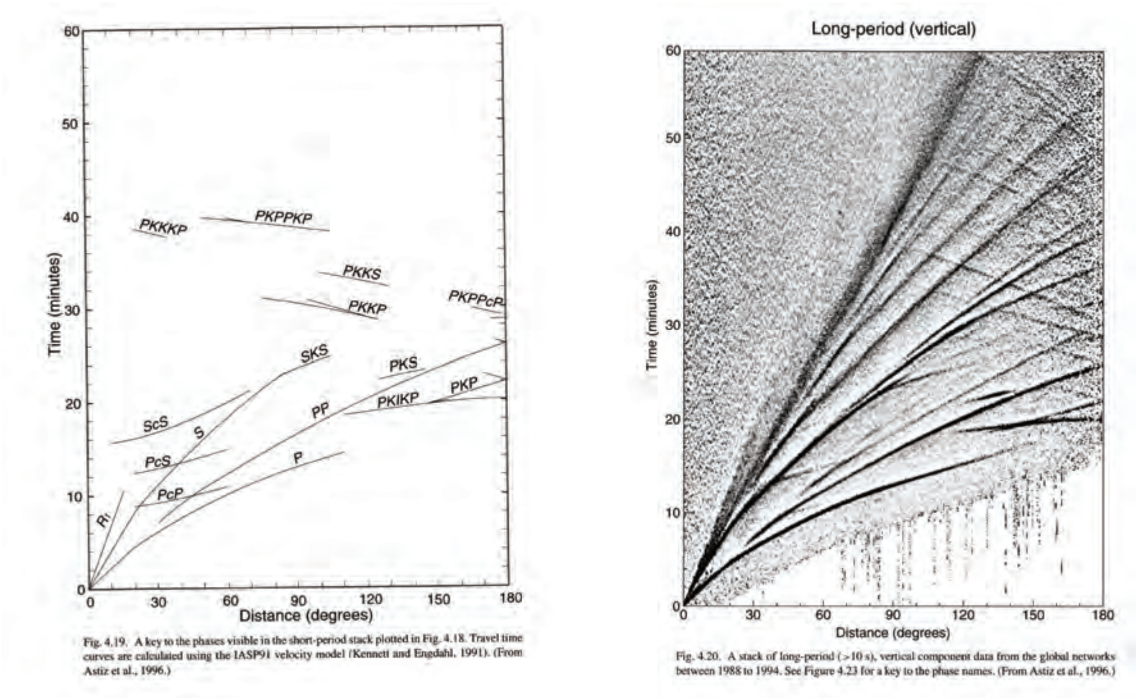


Figure 4.1 (from Shearer, P.M., 1999)

We try to find the waves

- propagating in the horizontal direction
- satisfying both the free surface conditions and the elastic equation of motion

Rayleigh Waves in a Homogeneous Halfspace

Horizontally propagating waves:

$$\begin{aligned} \gamma &= A \exp\left(i\left(\omega t - k_x x\right) - k_{z\alpha} z\right) \\ \Psi &= B \exp\left(i\left(\omega t - k_x x\right) - k_{z\beta} z\right) \end{aligned}$$

Free surface boundary conditions:

$$\mu\left\{2\left(-ik_x\right)\left(-k_{z\alpha}\right)A - \left(-k_{z\beta}\right)\left(-k_{z\beta}\right)B + \left(-ik_x\right)\left(-ik_x\right)B\right\} = 0$$

$$\lambda \left\{ (-ik_x)(-ik_x)A + (-k_{z\alpha})(-k_{z\alpha})A \right\} \\ +2\mu \left\{ (-k_{z\alpha})(-k_{z\alpha})A + (-ik_x)(-k_{z\beta})B \right\} = 0$$

$$\begin{pmatrix} 2ik_x k_{z\alpha}\,\mu & -\left({k_x}^2 + {k_{z\beta}}^2\right)\mu \\ -{k_x}^2\lambda + {k_{z\alpha}}^2(\lambda + 2\mu) & 2ik_x k_{z\beta}\,\mu \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To have non-trivial solutions, we need

$$\begin{vmatrix} 2ik_x k_{z\alpha}\,\mu & -\left({k_x}^2 + {k_{z\beta}}^2\right)\mu \\ -{k_x}^2\lambda + {k_{z\alpha}}^2(\lambda + 2\mu) & 2ik_x k_{z\beta}\,\mu \end{vmatrix} = 0.$$

After tedious mathematics, we have

$$\left(2 - \frac{{c_x}^2}{\beta^2}\right)^2 - 4\left(1 - \frac{{c_x}^2}{\alpha^2}\right)^{1/2}\left(1 - \frac{{c_x}^2}{\beta^2}\right)^{1/2} = 0.$$

For the medium with $\lambda = \mu$ (i.e. $\alpha = \sqrt{3}\beta$), we have

$$c_x = \frac{2}{\sqrt{3 + \sqrt{3}}} = 0.9194\beta$$

The apparent velocity of the Rayleigh wave is slightly less than the shear velocity.
(for a homogeneous halfspace with $\lambda = \mu$)

Rayleigh Wave Displacements

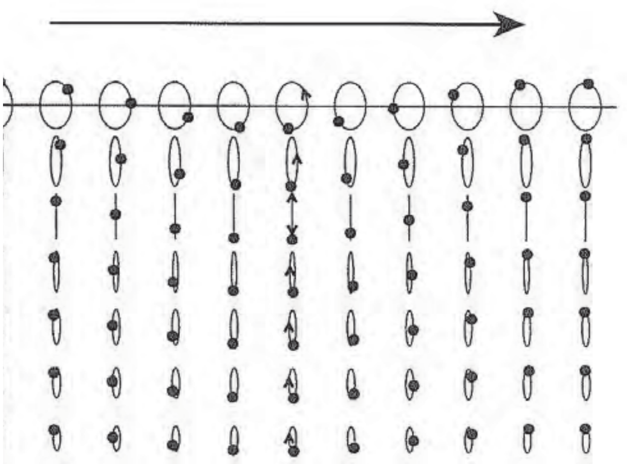


Figure 4.2 (form Shearer, P.M., 1999)

Type of Surface Wave

P-SV surface wave: Rayleigh wave
SH surface wave: Love wave

Generally speaking,
Love wave speed > Rayleigh wave speed

Dispersion

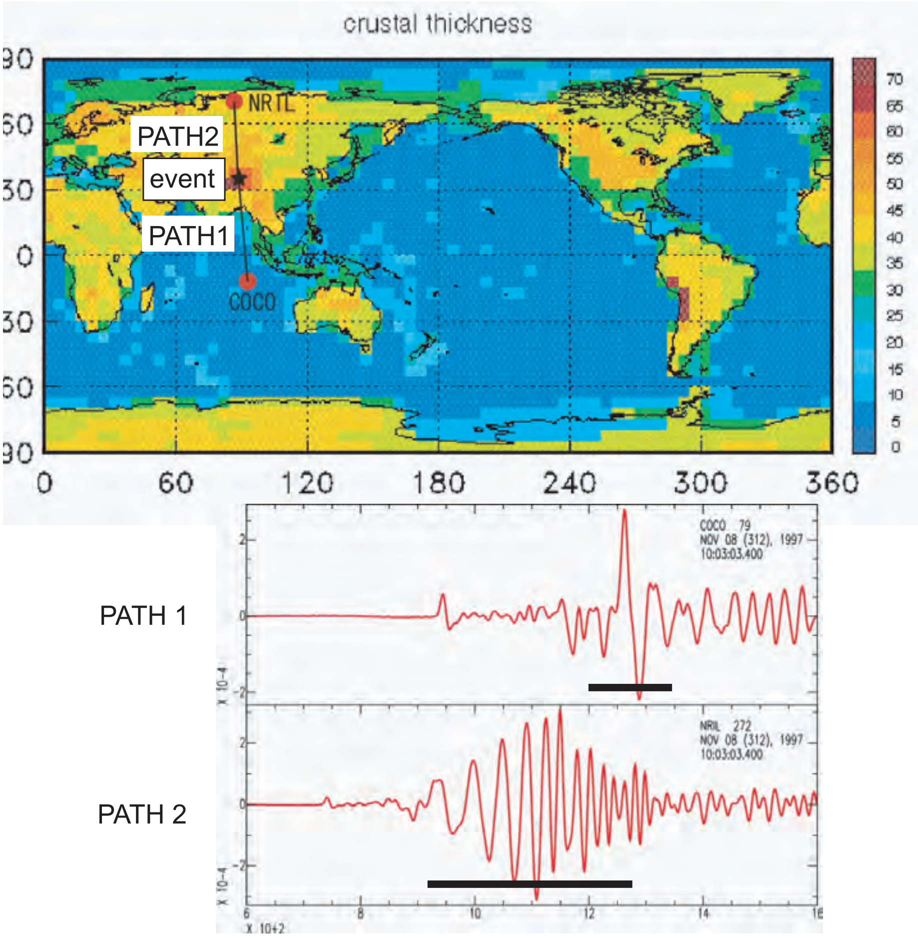


Figure 4.3

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