Basic Theory

for CCA Method

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IISEE, BRI, Japan

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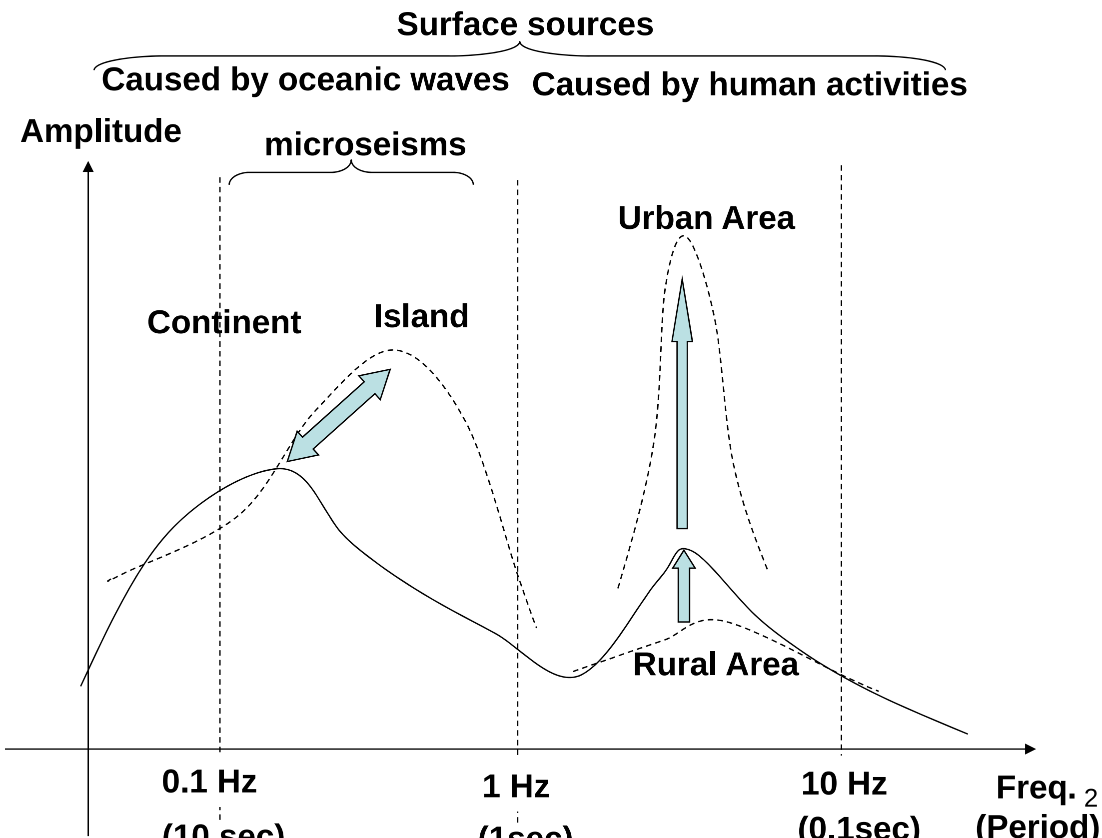
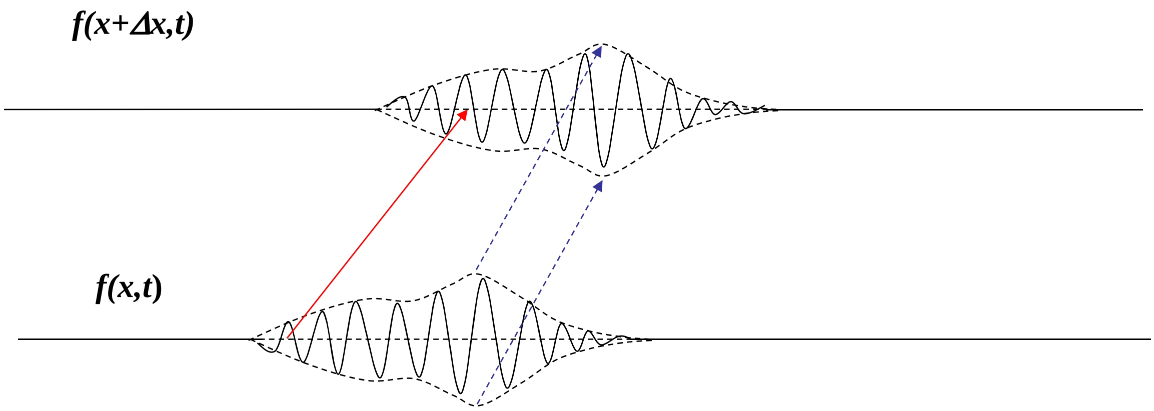
Microtremor (Ambient Noise)



Preparation: Phase Velocity & Group Velocity



Propagation of Energy: Group Velocity *v*.

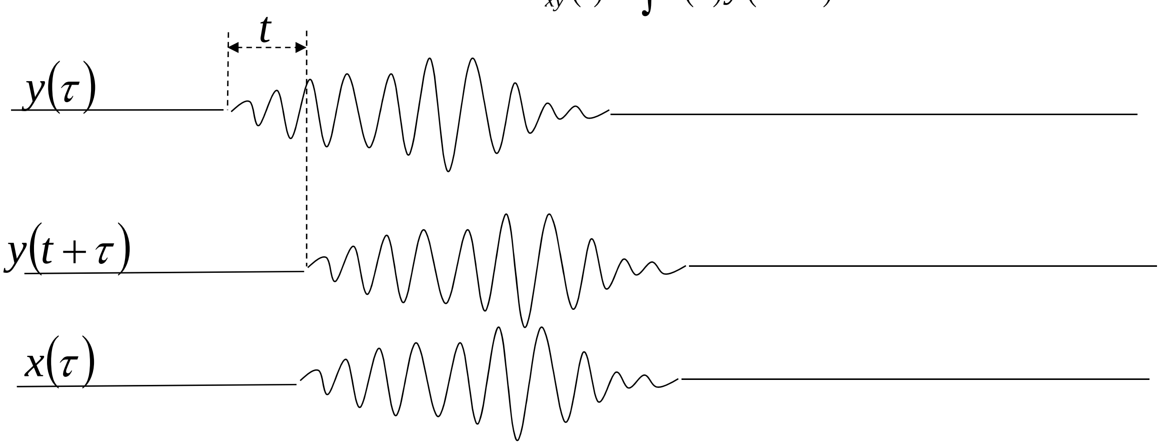
Propagation of Information: Phase Velocity *c*.

$$v = c + \frac{dv}{dk} \qquad k = \frac{\omega}{c}$$

Dependency of Phase Velocity on the Frequency *f* (or Wave Number *k*): Dispersion.
c is different from v in Dispersive Media.

Preparation: Phase Velocity Determination

Definition of Cross-correlation $C_{xy}(t) \equiv \int x(\tau) y(t+\tau) d\tau$

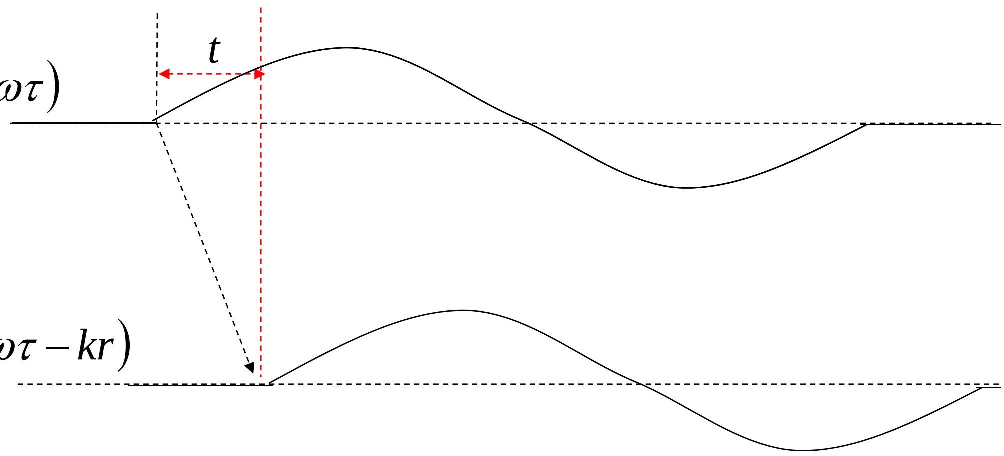


Product & Integration ⟶ **Cross-correlation**

Preparation: Phase Velocity Determination by Cross-Correlation

Cosine Function with the angular frequency *ω* (one cycle only)

$$f(0,\tau) = f_0 \cos(\omega\tau)$$



$$f(r,\tau) = f_0 \cos(\omega\tau - kr)$$

Time lag *t* gives coincidence. If distance is *r*, the phase velocity *c* is given by *r/t*.

$$
\begin{aligned}
Cc(0,r,t) &= f_0^2 \int \cos(\omega\tau)\cos(\omega\tau + \omega t - kr)\,d\tau \\
&= f_0^2 \left[\int \cos^2(\omega\tau)\,d\tau \cos(\omega t - kr) - \int \cos(\omega\tau)\sin(\omega\tau)\,d\tau \sin(\omega t - kr)\right] \\
&\propto f_0^2 \cos(\omega t - kr)
\end{aligned}
$$

The maximum value of *Cc* corresponds to this time lag.

In the frequency domain:

$$Cc(0,r,\omega) = F(0,\omega)\cdot\overline{F(r,\omega)} = |F(0,\omega)|\cdot|F(r,\omega)|\cdot\exp(i\Delta\phi(\omega))$$

Phase lag due to wave propagation is

$$\Delta\phi = \frac{\omega r}{c} \qquad \textbf{because} \qquad \exp\left\{i\omega\left(t + \frac{r}{c}\right)\right\} = \exp\left\{i\left(\omega t + \frac{\omega r}{c}\right)\right\}$$

Therefore,

$$Cc(0,r,\omega) = |F(0,\omega)|\cdot|F(r,\omega)|\cdot\exp\left(i\frac{\omega r}{c}\right)$$

Coherence

$$Coh(0,r,\omega) = \mathrm{Re}\left[\frac{Cc(0,r,\omega)}{|F(0,\omega)|\cdot|F(r,\omega)|}\right] = \mathrm{Re}\left[\exp\left(i\frac{\omega r}{c}\right)\right] = \cos\left(\frac{\omega r}{c}\right)$$

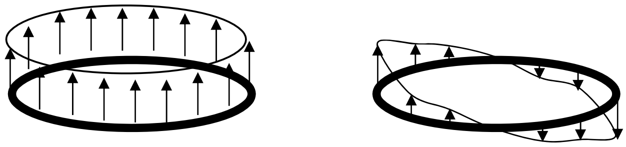
Here, *c* is the phase velocity measured along the measurement line.

Auto-Correlation

$$Ac(0,\omega) = Cc(0,0,\omega) = |F(0,\omega)|^2,\ Ac(r,\omega) = Cc(r,r,\omega) = |F(r,\omega)|^2$$

What's CCA ?

Zero and 1st order Fourier transforms of the wave field along the circle over the azimuth.



Zero Order **1st Order**

CCA (Cho et al., 2006, Tada et al., 2007)

$$s_{CCA}(r,\omega)=\frac{PSD\left\langle \int_{-\pi}^{\pi} Z(t,r,\theta)d\theta \right\rangle}{PSD\left\langle \int_{-\pi}^{\pi} Z(t,r,\theta)\exp(-i\theta)d\theta \right\rangle}=\frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$

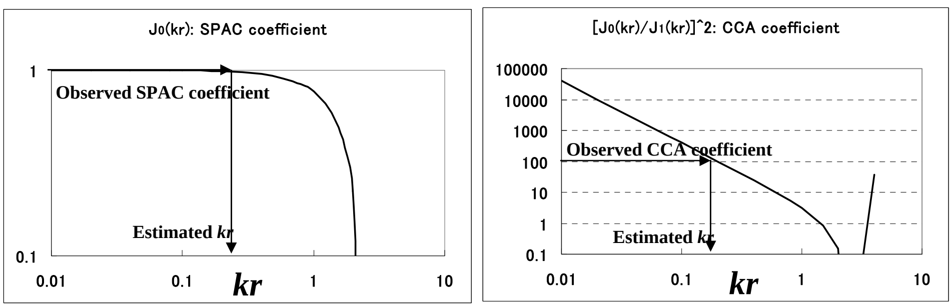
PSD<> denotes the power spectral density

SPAC (Aki, 1957;1965; Okada et al. 1987; Okada, 2003)

$$\rho(r,\omega)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{Re}\{E[C_{c,m}(r,\omega)]\}}{\sqrt{E[C_{c,c}(\omega)]E[C_{m,m}(\omega)]}}d\theta=J_0(kr)$$

CCA (Cho et al., 2006, Tada et al., 2007)

$$s_{CCA}(r,\omega)=\frac{PSD\left\langle \int_{-\pi}^{\pi} Z(t,r,\theta)d\theta \right\rangle}{PSD\left\langle \int_{-\pi}^{\pi} Z(t,r,\theta)\exp(-i\theta)d\theta \right\rangle}=\frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$



Schematic graphs for comparison of SPAC with CCA.

Analysis in the frequency domain

$$G_{Z0Z0}(r,r,\omega) = PSD\left\langle \int_{-\pi}^{\pi} Z(t,r,\theta)d\theta \right\rangle$$

$$\approx PSD\left\langle \left\{\frac{2\pi}{M}\sum_{m=1}^{M} Z(\omega,r,\theta_m)\right\}\left\{\frac{2\pi}{M}\sum_{m'=1}^{M} Z(\omega,r,\theta_{m'})^*\right\}\right\rangle = \frac{4\pi^2}{M^2}\sum_{m=1}^{M}\sum_{m'=1}^{M} E\left[C_{m,m'}(r,\omega)\right]$$

$$G_{Z1Z1}(r,r,\omega) = PSD\left\langle \int_{-\pi}^{\pi} Z(t,r,\theta)\exp(-i\theta)d\theta \right\rangle$$

$$\approx PSD\left\langle \left\{\frac{2\pi}{M}\sum_{m=1}^{M} Z(\omega,r,\theta_m)\exp(-i\theta_m)\right\}\left\{\frac{2\pi}{M}\sum_{m'=1}^{M} Z(\omega,r,\theta_{m'})\exp(-i\theta_{m'})\right\}\right\rangle^*$$

$$= \frac{4\pi^2}{M^2}\sum_{m=1}^{M}\sum_{m'=1}^{M} E\left[C_{m,m'}(r,\omega)\right]\exp\{-i(\theta_m - \theta_{m'})\}$$

PSD: Power Spectral Density, $C_{mm'}(r,\omega)$: Cross correlation

Analysis in the frequency domain

CCA coefficient

$$s_{CCA'}(r,\omega) \approx \frac{\sum_{m=1}^{M}\sum_{m'=1}^{M} E\left[C_{m,m'}(r,\omega)\right]}{\sum_{m=1}^{M}\sum_{m'=1}^{M} E\left[C_{m,m'}(r,\omega)\right]\exp\{-i(\theta_m - \theta_{m'})\}} \approx \frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$

Analysis in the frequency domain

$$G_{Z0Z1}(r,r,\omega) = CSD\left\langle \int_{-\pi}^{\pi} Z(t,r,\theta)d\theta, \int_{-\pi}^{\pi} Z(t,r,\theta)\exp(-i\theta)d\theta \right\rangle$$

$$\approx E\left[\left\{\frac{2\pi}{M}\sum_{m'=1}^{M} Z(\omega,r,\theta_{m'})\right\}\left\{\frac{2\pi}{M}\sum_{m=1}^{M} Z(\omega,r,\theta_m)\exp(-i\theta_m)\right\}^*\right]$$

$$= \frac{4\pi^2}{M^2}\sum_{m=1}^{M}\sum_{m'=1}^{M} E\left[C_{m,m'}(r,\omega)\right]\exp(-i\theta_m)$$

Incoming azimuth

$$\phi_R = Arg\left[G_{Z0Z1}(r,r,\omega)\right] - \frac{\pi}{2}$$

CSD: Density of Cross Spectra

$$G_{Z0Z0}(0,r,\omega) = CSD\left\langle \int_{-\pi}^{\pi} Z(t,0,0)d\theta, \int_{-\pi}^{\pi} Z(t,r,\theta)d\theta \right\rangle$$

$$= 2\pi E\left[Z(\omega,0,0)\left\{\int_{-\pi}^{\pi} Z(\omega,r,\theta)d\theta\right\}^*\right] = 2\pi E\left[\int_{-\pi}^{\pi} Z(\omega,0,0)Z(\omega,r,\theta)^* d\theta\right]$$

$$= 2\pi E\left[\int_{-\pi}^{\pi} C_{0,\theta}(r,\omega)d\theta\right] \approx \frac{4\pi^2}{M}\sum_{m=1}^{M} E\left[C_{0,m}(r,\omega)\right]$$

$$G_{Z0Z0}(0,0,\omega) = CSD\left\langle \int_{-\pi}^{\pi} Z(t,0,0)d\theta, \int_{-\pi}^{\pi} Z(t,0,0)d\theta \right\rangle$$

$$= 4\pi^2 E\left[Z(\omega,0,0)Z(\omega,0,0)^*\right] = 4\pi^2 E\left[C_{0,0}(r,\omega)\right]$$

SPAC coefficient

$$\rho(r,\omega) = \frac{1}{2\pi}\frac{\int_{-\pi}^{\pi} E\left[C_{0,\theta}(r,\omega)\right]d\theta}{E\left[C_{0,0}(r,\omega)\right]} = \frac{G_{Z0Z0}(0,r,\omega)}{G_{Z0Z0}(0,0,\omega)}$$

$$\approx \frac{1}{M}\frac{\sum_{m=1}^{M} E\left[C_{0,m}(r,\omega)\right]}{E\left[C_{0,0}(0,\omega)\right]} = J_0(r\omega/c)$$

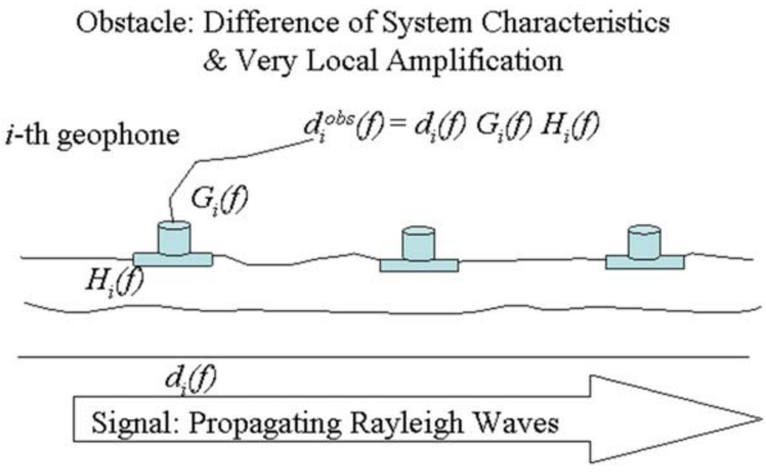
Correction for incoherent noise (Cho et al. 2006)

$$s_{CCA}(r,\omega)=\frac{J_0^2(r\omega/c)+\varepsilon(\omega)/N}{J_1^2(r\omega/c)+\varepsilon(\omega)/N}$$

$$\begin{aligned}
coh^2(\omega)&=\frac{\left|G_{Z0Z0}(0,r,\omega)\right|^2}{G_{Z0Z0}(r,r,\omega)G_{Z0Z0}(0,0,\omega)},\\
A(\omega)&=-\rho^2(\omega)^2,\\
B(\omega)&=\frac{\rho^2(\omega)}{coh^2(\omega)}-2\rho^2(\omega)-\frac{1}{N},\\
C(\omega)&=\rho^2(\omega)\left\{\frac{1}{coh^2(\omega)}-1\right\},\\
\varepsilon(\omega)&=\left\{-B(\omega)-\sqrt{B^2(\omega)-4A(\omega)C(\omega)}\right\}/2A(\omega).
\end{aligned}$$

System Correction for CCA in the frequency domain.

Yokoi (2012)



Record of *i*-th channnel

Ground motion of propagating waves

$$d_i^{obs}(f)=d_i(f)\cdot G_i(f)\cdot H_i(f)$$

System Characteristics

Very Local Effect

$$G_i(f)=\left|G_i(f)\right|\exp\{-j\varphi_i\} \qquad H_i(f)=\left|H_i(f)\right|\exp\{-j\phi_i\}$$

Cross spectra between *i*-th and *k*-th channnels' records

$$C_{ik}^{obs}(f) = d_i^{obs}(f) \cdot d_k^{obs}(f)^*$$
$$= C_{ik}(f) \cdot G_i(f) \cdot G_k(f)^* \cdot H_i(f) \cdot H_k(f)^*$$

Cross spectra between i-th and k-th channnels' ground motion

CCA coefficient is defined using cross spectra of ground motion

$$s_{CCA}(r,\omega) = \frac{\sum_{i=1}^{M}\sum_{k=1}^{M} E[C_{ik}(\omega)]}{\sum_{i=1}^{M}\sum_{k=1}^{M} E[C_{ik}(\omega)]\exp\{-j(\theta_i - \theta_k)\}}$$

Phase difference among the system characteristics

$$\exp\{-j(\varphi_i - \varphi_k)\} = \frac{G_i(\omega)\cdot G_k(\omega)^*}{|G_i(\omega)||G_k(\omega)|} = \frac{|d^{huddle}(\omega)|^2 G_i(\omega)\cdot G_k(\omega)^*}{|d^{huddle}(\omega)|^2 |G_i(\omega)||G_k(\omega)|}$$
$$= \frac{|d^{huddle}(\omega)|^2 G_i(\omega)\cdot G_k(\omega)^*}{\sqrt{|d^{huddle}(\omega)|^2 |G_i(\omega)|^2 |d^{huddle}(\omega)|^2 |G_k(\omega)|^2}}$$
$$= \frac{C_{ik}^{huddle}(\omega)}{\sqrt{C_{ii}^{huddle}(\omega)\cdot C_{kk}^{huddle}(\omega)}}$$

Correction factor for phase difference among the system characteristics

$$Cor_{ik}^{huddle}(\omega) = \exp\{j(\varphi_i - \varphi_k)\} = \frac{\sqrt{C_{ii}^{huddle}(\omega)\cdot C_{kk}^{huddle}(\omega)}}{C_{ik}^{huddle}(\omega)}$$

For a stable processing, average over time blocks is applied

$$\overline{Cor_{ik}^{huddle}(f)} = \exp\left\{ jE\left[Arg\left(\frac{\sqrt{C_{ii}^{huddle}(f)\cdot C_{kk}^{huddle}(f)}}{C_{ik}^{huddle}(f)} \right)\right]\right\}$$

Corrected cross spectra

$$\begin{aligned} C_{ik}^{cor}(\omega) &= C_{ik}^{obs}(\omega)\cdot\overline{Cor_{ik}^{huddle}(\omega)} \\ &= C_{ik}(\omega)\left|G_i(\omega)\right|\left|G_k(\omega)\right|H_i(\omega)H_k(\omega)^* \end{aligned}$$

Complex coherence between *i*-th and *k*-th channels' corrected records

$$Coh_{ik}^{cor}(\omega) = \frac{E\left[C_{ik}^{cor}(\omega)\right]}{\sqrt{E\left[C_{ii}^{cor}(\omega)\right]E\left[C_{kk}^{cor}(\omega)\right]}} = \frac{E\left[C_{ik}(\omega)\right]}{\sqrt{E\left[C_{ii}(\omega)\right]E\left[C_{kk}(\omega)\right]}}\cdot\exp\left\{-j(\phi_i-\phi_k)\right\}$$

An interim quantity $R_{ik}(\omega)$

$$R_{ik}(\omega) = C_{00}^{cor}(\omega)\cdot Coh_{ik}^{cor}(\omega) = C_{00}^{cor}(\omega)\cdot\frac{E\left[C_{ik}^{cor}(\omega)\right]}{\sqrt{E\left[C_{ii}^{cor}(\omega)\right]E\left[C_{kk}^{cor}(\omega)\right]}}$$

$$= C_{00}^{obs}(\omega)\cdot\frac{E\left[C_{ik}^{obs}(\omega)\right]\overline{Cor_{ik}^{huddle}(\omega)}}{\sqrt{E\left[C_{ii}^{obs}(\omega)\right]\overline{Cor_{ii}^{huddle}(\omega)}E\left[C_{kk}^{obs}(\omega)\right]\overline{Cor_{kk}^{huddle}(\omega)}}}$$

$$= \frac{C_{00}^{obs}(\omega)E\left[C_{ik}^{obs}(\omega)\right]\overline{Cor_{ik}^{huddle}(\omega)}}{\sqrt{E\left[C_{ii}^{obs}(\omega)\right]E\left[C_{kk}^{obs}(\omega)\right]}}$$

:Description using observed records

On the other hand,

$$R_{ik}(\omega) = C_{00}^{obs}(\omega)\cdot\frac{E\left[C_{ik}(\omega)\right]}{\sqrt{E\left[C_{ii}(\omega)\right]E\left[C_{kk}(\omega)\right]}}\cdot\exp\left\{-j(\phi_i-\phi_k)\right\}$$

Assumptions:

1) Phase difference due to very local effect is negligible $\phi_i - \phi_k \approx 0$

2) Power spectra of ground motion is same for all channels

$$P(\omega) = E[C_{ii}(\omega)] = E[C_{kk}(\omega)]$$

$$R_{ik}(\omega) \approx \left(\frac{C_{00}^{obs}(\omega)}{P(\omega)} \right) E[C_{ik}(\omega)]$$

Contents of ( ) is common for all channel pairs, then

$$s_{CCA}(\omega) = \frac{\sum_{i=1}^{M}\sum_{k=1}^{M} E[C_{ik}(\omega)]}{\sum_{i=1}^{M}\sum_{k=1}^{M} E[C_{ik}(\omega)]\exp\{-j(\theta_i - \theta_k)\}} \approx \frac{\sum_{i=1}^{M}\sum_{k=1}^{M} R_{ik}(\omega)}{\sum_{i=1}^{M}\sum_{k=1}^{M} R_{ik}(\omega)\exp\{-j(\theta_i - \theta_k)\}}$$

Complete definition of CCA coefficient is obtained by using $R_{ik}(\omega)$ in place of $C_{ik}^{obs}(\omega)$

SPAC coefficient (Aki 1957, Okada 2003)

$$\rho(r,\omega) \approx \frac{1}{M} \frac{\sum_{m=1}^{M} E[C_{0,m}(r,\omega)]}{E[C_{0,0}(0,\omega)]} = J_0(r\omega / c)$$

Use $R_{0,m}(\omega)$ in place of $E[C_{0,m}(\omega)]$ $\quad R_{ik}(\omega) = \dfrac{C_{00}^{obs}(\omega) E[C_{ik}^{obs}(\omega)] \overline{Cor_{ik}^{huddle}(\omega)}}{\sqrt{E[C_{ii}^{obs}(\omega)] E[C_{kk}^{obs}(\omega)]}}$

Neglect phase difference of system characteristic

$$\overline{Cor_{ik}^{huddle}(\omega)} = \exp\{jE[\varphi_i - \varphi_k]\} \approx 1$$

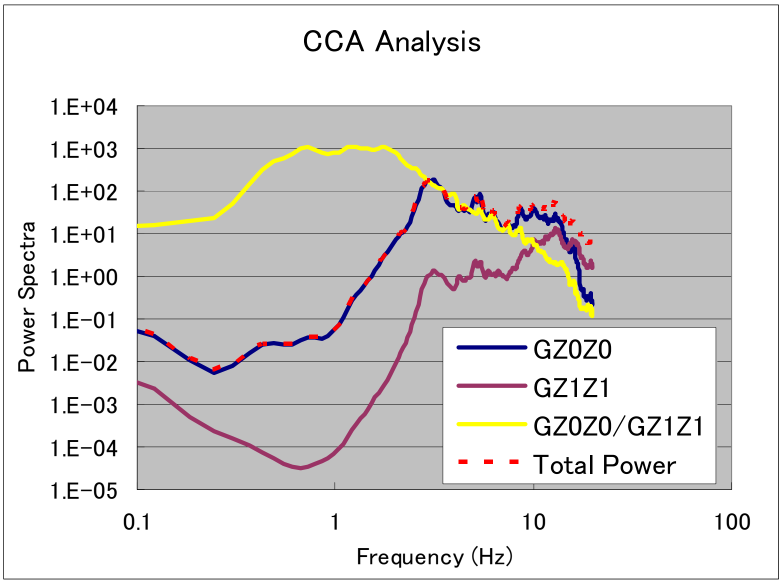
Or perform system correction

$$E[C_{ik}^{obs}(\omega)] \overline{Cor_{ik}^{huddle}(\omega)} \Rightarrow E[C_{ik}^{obs}(\omega)]$$

Okada et al. (1987)'s formula is derived by two assumptions in the previous slide

$$\rho(r,\omega) \approx \frac{1}{M} \frac{\sum_{m=1}^{M} E[C_{0,m}(r,\omega)]}{E[C_{0,0}(0,\omega)]} = \frac{1}{M} \frac{\sum_{m=1}^{M} R_{0,m}(\omega)}{R_{0,0}(\omega)} \approx \frac{1}{M} \sum_{m=1}^{M} \frac{E[C_{0,m}^{obs}(\omega)]}{\sqrt{E[C_{0,0}^{obs}(\omega)] E[C_{m,m}^{obs}(\omega)]}}$$

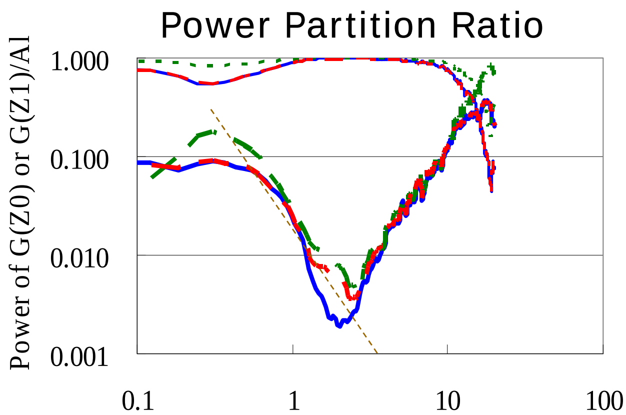
$$s_{CCA}(r,\omega) = \frac{G_{Z0Z0}(r,r,\omega)}{G_{Z1Z1}(r,r,\omega)} = \frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$

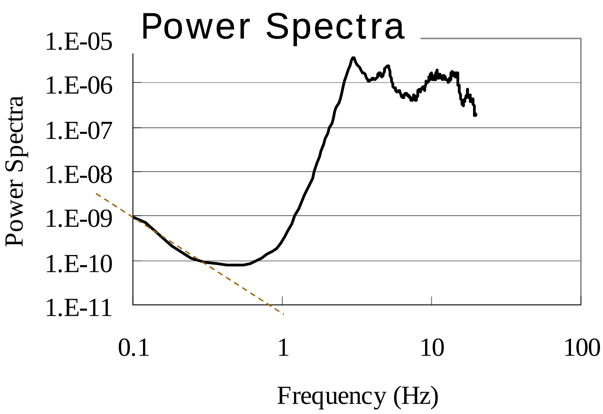


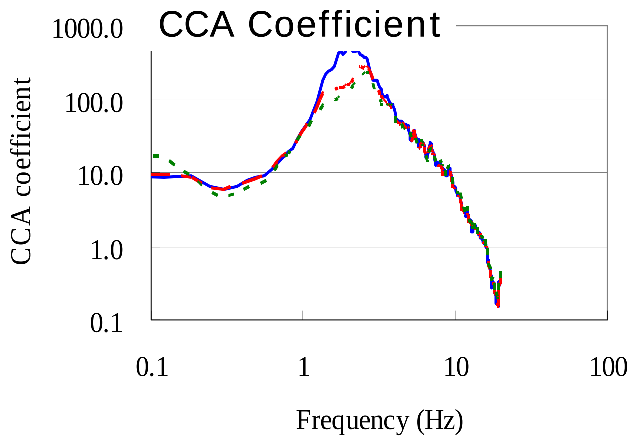
A real example of CCA Analysis
(BRI, Tsukuba, Japan)

CCA

Example using 2Hz seismometers



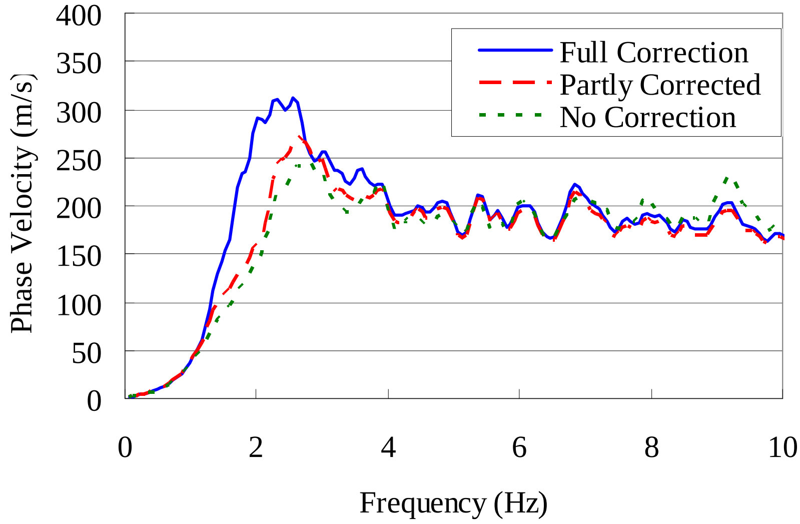




System Correction

Phase velocity

Effect of System Correction

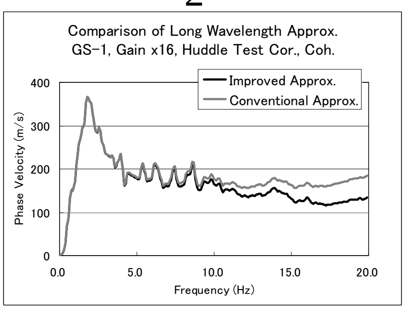


$$s(r,\omega)=\frac{J_0^2(r\omega/c)}{J_1^2(r\omega/c)}$$

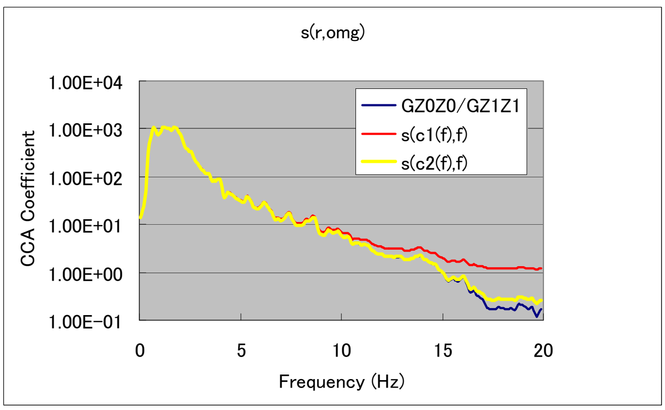
Long wavelength apploximation
(Small value of *kr*) (Cho et al,2006)

$$c(\omega)=\frac{r\omega}{2}\sqrt{s(r,\omega)+2}$$

c1



Improved approximation:
Yokoi(2012)



c2

$$c(\omega)/r\omega=1.0003\left(\frac{\sqrt{s(r,\omega)+0.97221}}{2.0003}+0.0015245\right)-0.0138$$

Dispersion Curve

Heuristic Search: VFSA-DHSM

Vs Structure