Basic Theory
for SPAC Method

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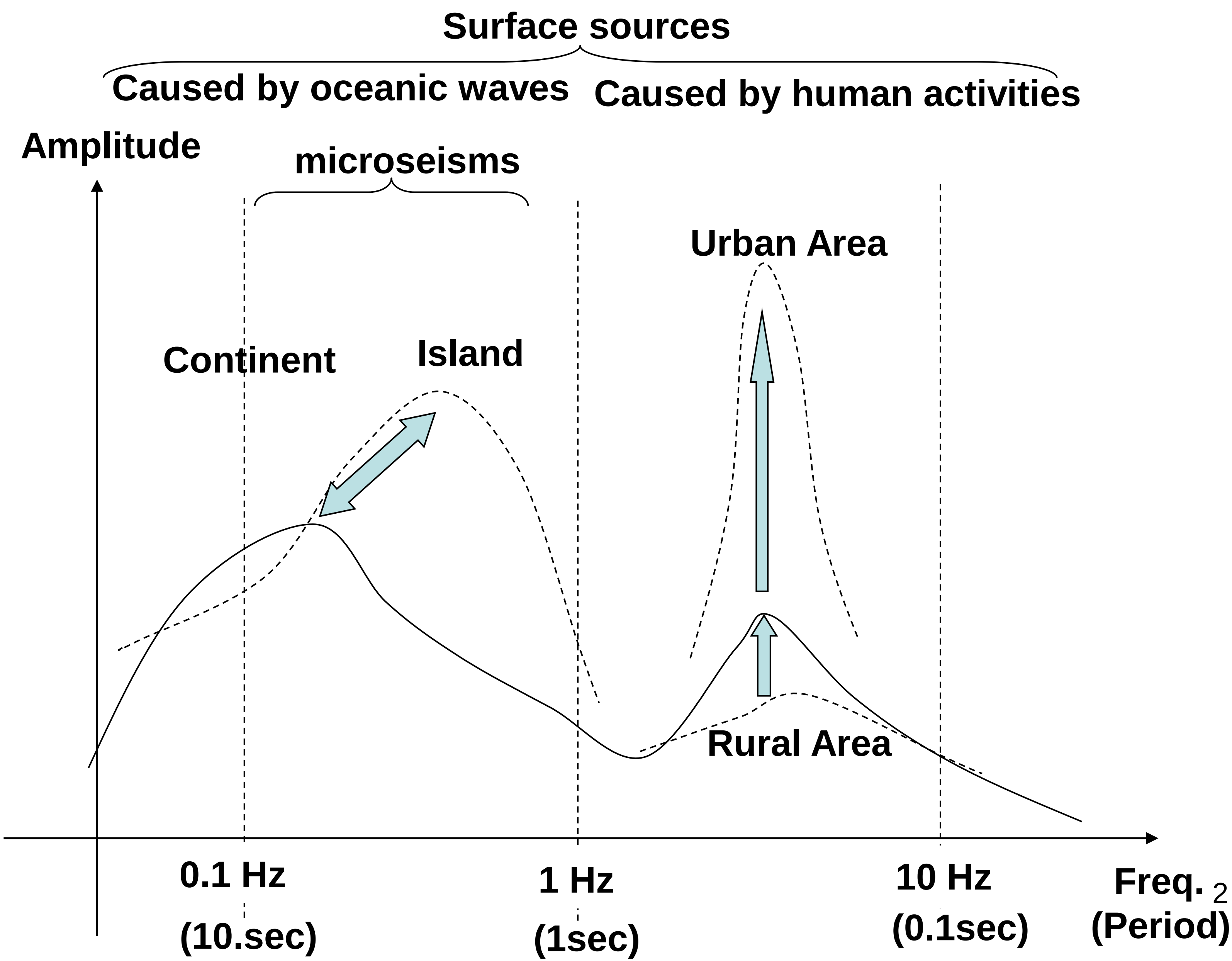
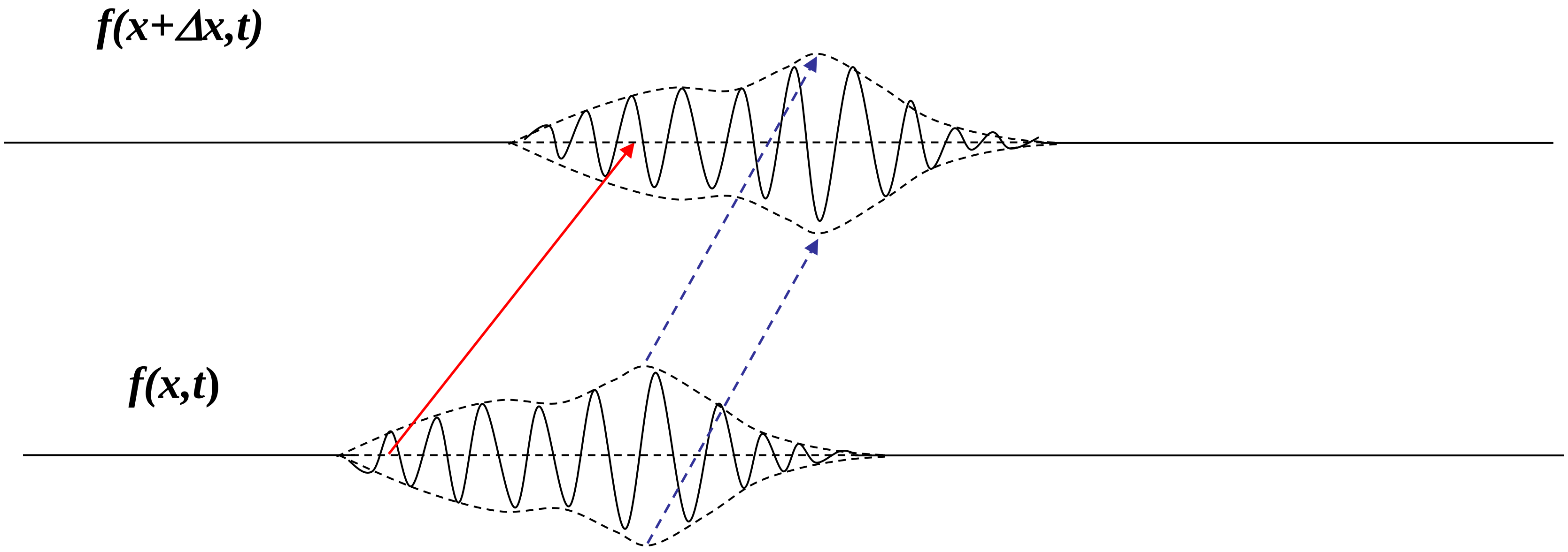
Pre-Symposium Training Course
7-th General Assembly of Asian Seismological Commission

Microtremor (Ambient Noise)



Preparation: Phase Velocity & Group Velocity



Propagation of Energy: Group Velocity *v*.

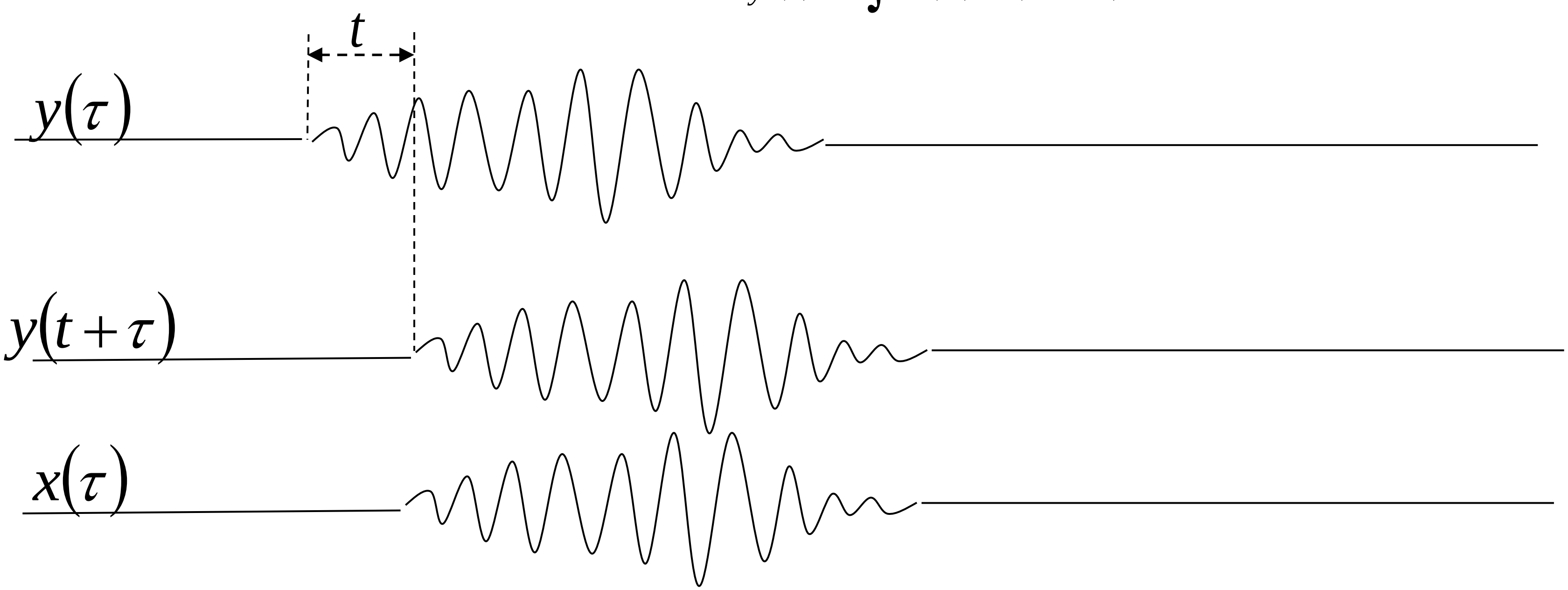
Propagation of Information: Phase Velocity *c*.

$$v = c + \frac{dv}{dk} \qquad k = \frac{\omega}{c}$$

Dependency of Phase Velocity on the Frequency *f* (or Wave Number *k*): Dispersion.
c is different from v in Dispersive Media.

Preparation: Phase Velocity Determination

Definition of Cross-correlation $C_{xy}(t) \equiv \int x(\tau) y(t+\tau) d\tau$

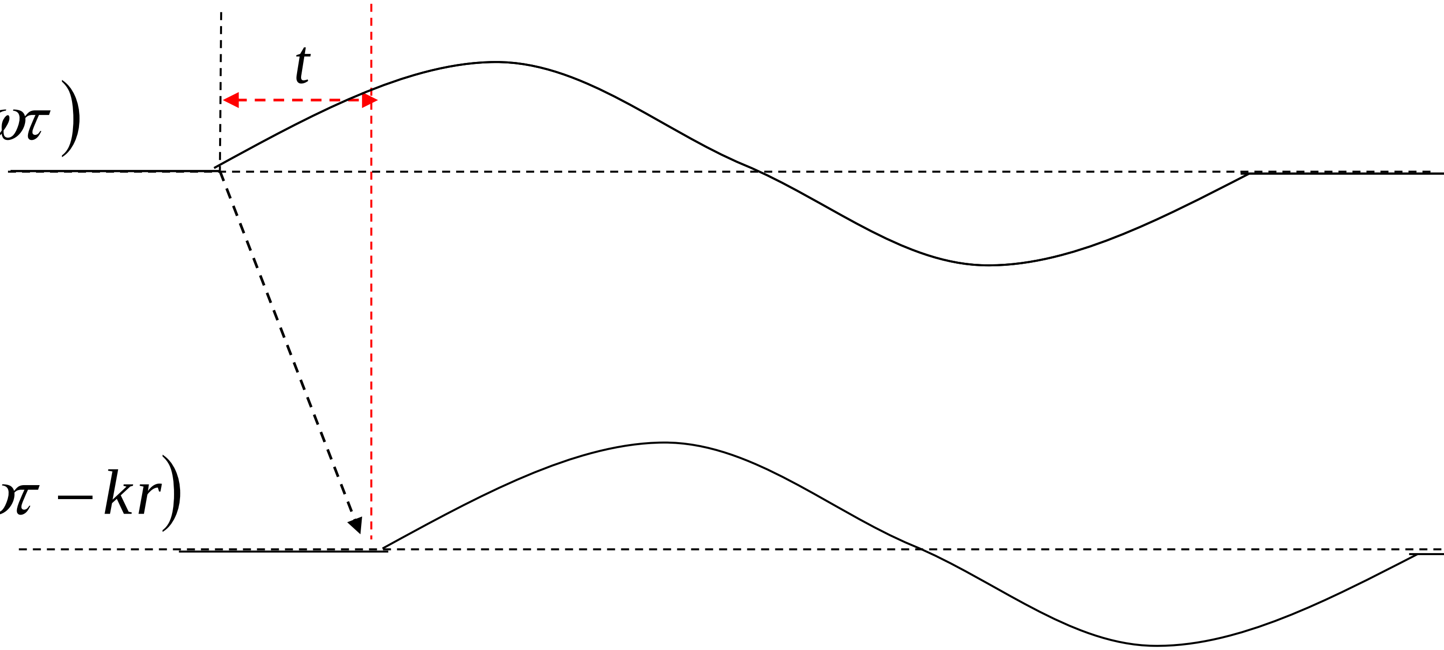


Product & Integration → **Cross-correlation**

Preparation: Phase Velocity Determination by Cross-Correlation

Cosine Function with the angular frequency ω (one cycle only)

$$f(0,\tau) = f_0 \cos(\omega\tau)$$



$$f(r,\tau) = f_0 \cos(\omega\tau - kr)$$

Time lag *t* gives coincidence. If distance is *r*, the phase velocity *c* is given by *r/t*.

$$\begin{aligned} Cc(0,r,t) &= f_0^2 \int \cos(\omega\tau)\cos(\omega\tau + \omega t - kr)\,d\tau \\ &= f_0^2\left[\int \cos^2(\omega\tau)\,d\tau \cos(\omega t - kr) - \int \cos(\omega\tau)\sin(\omega\tau)\,d\tau \sin(\omega t - kr)\right] \\ &\propto f_0^2 \cos(\omega t - kr) \end{aligned}$$

The maximum value of *Cc* corresponds to this time lag.

In the frequency domain:

$$Cc(0,r,\omega) = F(0,\omega)\cdot\overline{F(r,\omega)} = |F(0,\omega)|\cdot|F(r,\omega)|\cdot\exp(i\Delta\phi(\omega))$$

Phase lag due to wave propagation is

$$\Delta\phi = \frac{\omega r}{c} \qquad \textbf{because} \qquad \exp\left\{i\omega\left(t+\frac{r}{c}\right)\right\} = \exp\left\{i\left(\omega t+\frac{\omega r}{c}\right)\right\}$$

Therefore,

$$Cc(0,r,\omega) = |F(0,\omega)|\cdot|F(r,\omega)|\cdot\exp\left(i\frac{\omega r}{c}\right)$$

Coherence

$$Coh(0,r,\omega) = \mathrm{Re}\left[\frac{Cc(0,r,\omega)}{|F(0,\omega)|\cdot|F(r,\omega)|}\right] = \mathrm{Re}\left[\exp\left(i\frac{\omega r}{c}\right)\right] = \cos\left(\frac{\omega r}{c}\right)$$

Here, *c* is the phase velocity measured along the measurement line.

Auto-Correlation

$$Ac(0,\omega) = Cc(0,0,\omega) = |F(0,\omega)|^2,\ Ac(r,\omega) = Cc(r,r,\omega) = |F(r,\omega)|^2$$

Pioneering Work of Aki(1957)

Spatial Auto-Correlation (1D wave propagation along measurement line)

In the time domain

$$Cc(\Delta x, t) = f(x,t) * \overline{f(x+\Delta x, t)}$$

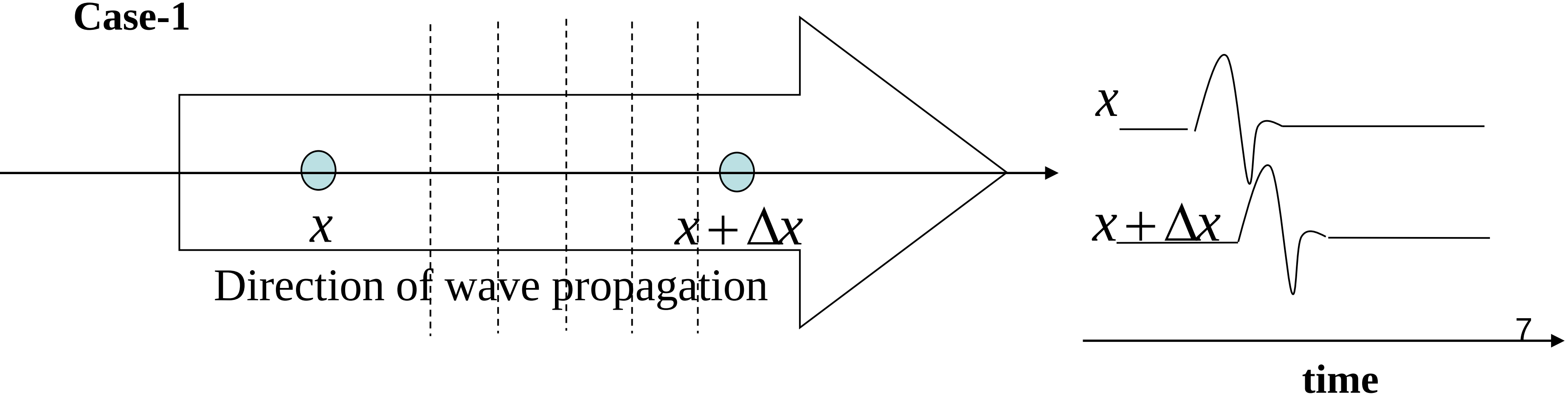
In the frequency domain

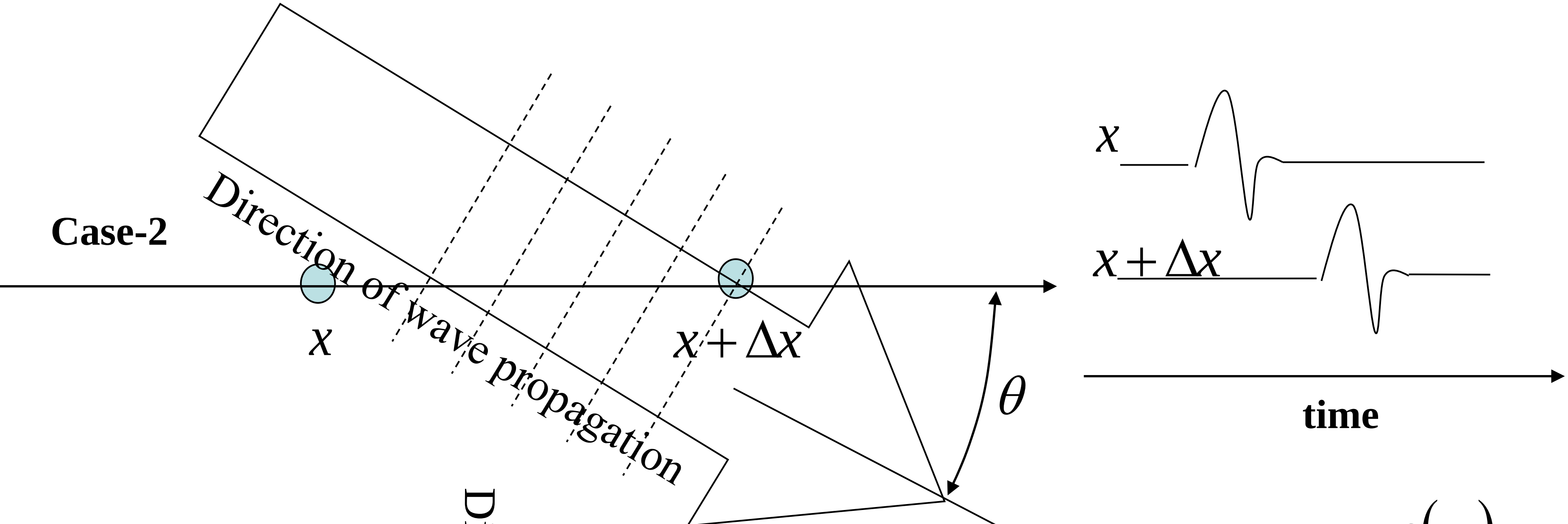
$$Cc(\Delta x, \omega) = F(x,\omega) \cdot \overline{F(x+\Delta x, \omega)} = |F(x,\omega)||F(x+\Delta x,\omega)| \exp\left(\frac{i\omega\Delta x}{c(\omega)}\right)$$

SPAC coefficient

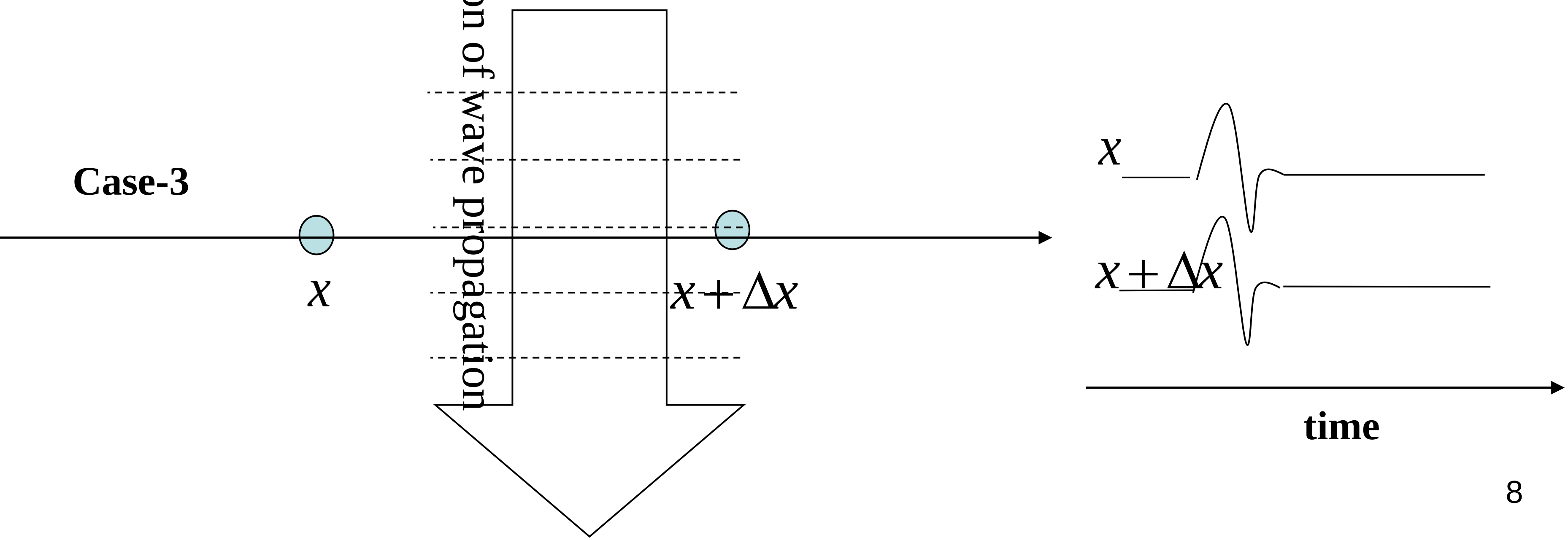
$$\rho(\Delta x, \omega) \equiv \mathrm{Re}\left[\frac{Cc(\Delta x,\omega)}{Cc(0,\omega)}\right] = \frac{|F(x+\Delta x,\omega)|}{|F(x,\omega)|}\mathrm{Re}\left[\exp\left(\frac{i\omega\Delta x}{c(\omega)}\right)\right] \approx \cos\left(\frac{\omega\Delta x}{c(\omega)}\right)$$

Case-1

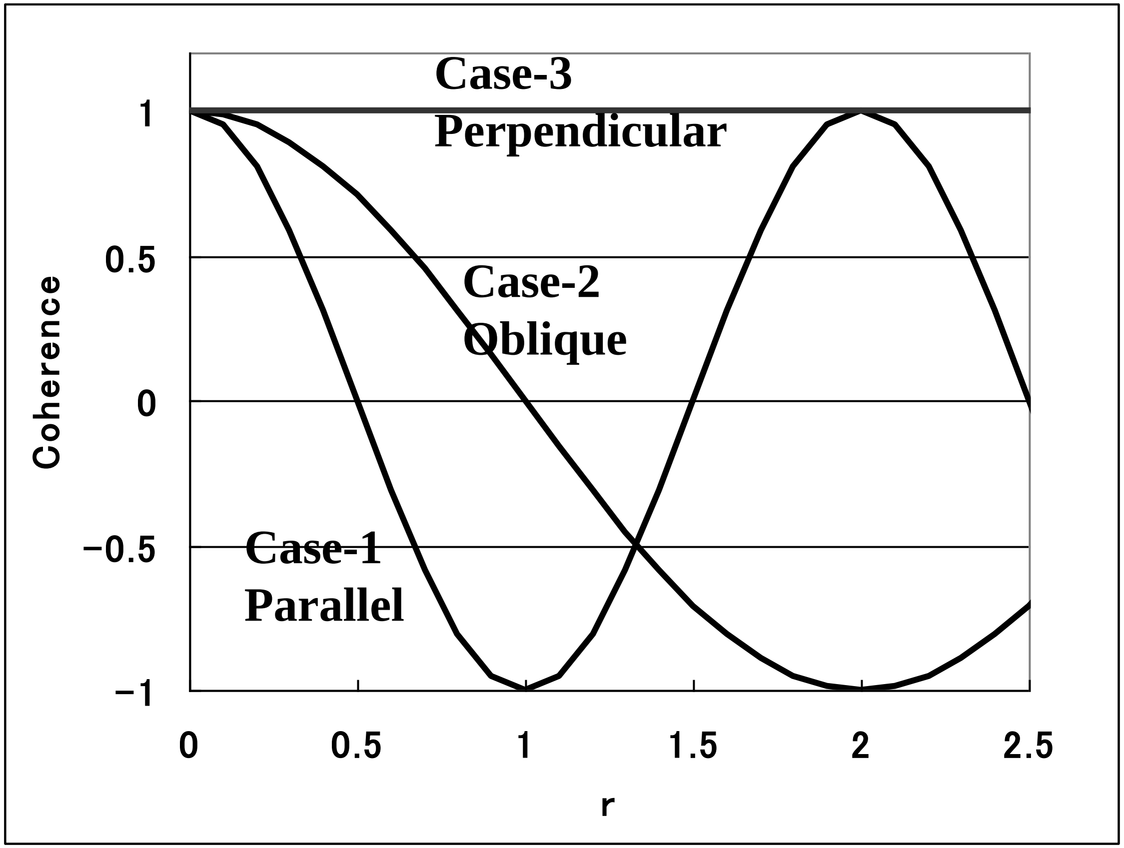




$$c_{apparent}(\omega,\theta)=\frac{c(\omega)}{\cos\theta}$$



Spatial Auto-Correlation (2D wave propagation toward one direction)



Spatial Auto-Correlation (2D wave propagation)

In the time domain

$$Cc(\xi,\eta,t) = f(x,y,t) * f(x+\xi, y+\eta, t)$$

In the frequency domain

$$Cc(\xi,\eta,\omega) = F(x,y,\omega)\cdot \overline{F(x+\xi, y+\eta, \omega)}$$

$$\xi = r\cos\psi,\ \eta = r\sin\psi$$

SPAC coefficient

$$\rho(r,\omega) = \frac{1}{2\pi}\int_0^{2\pi} Coh(\xi,\eta,\omega)\,d\psi$$

$$= \frac{1}{2\pi}\mathrm{Re}\left[\int_0^{2\pi}\exp\left(\frac{i\omega r}{c_{apparent}}\right)d\psi\right]$$

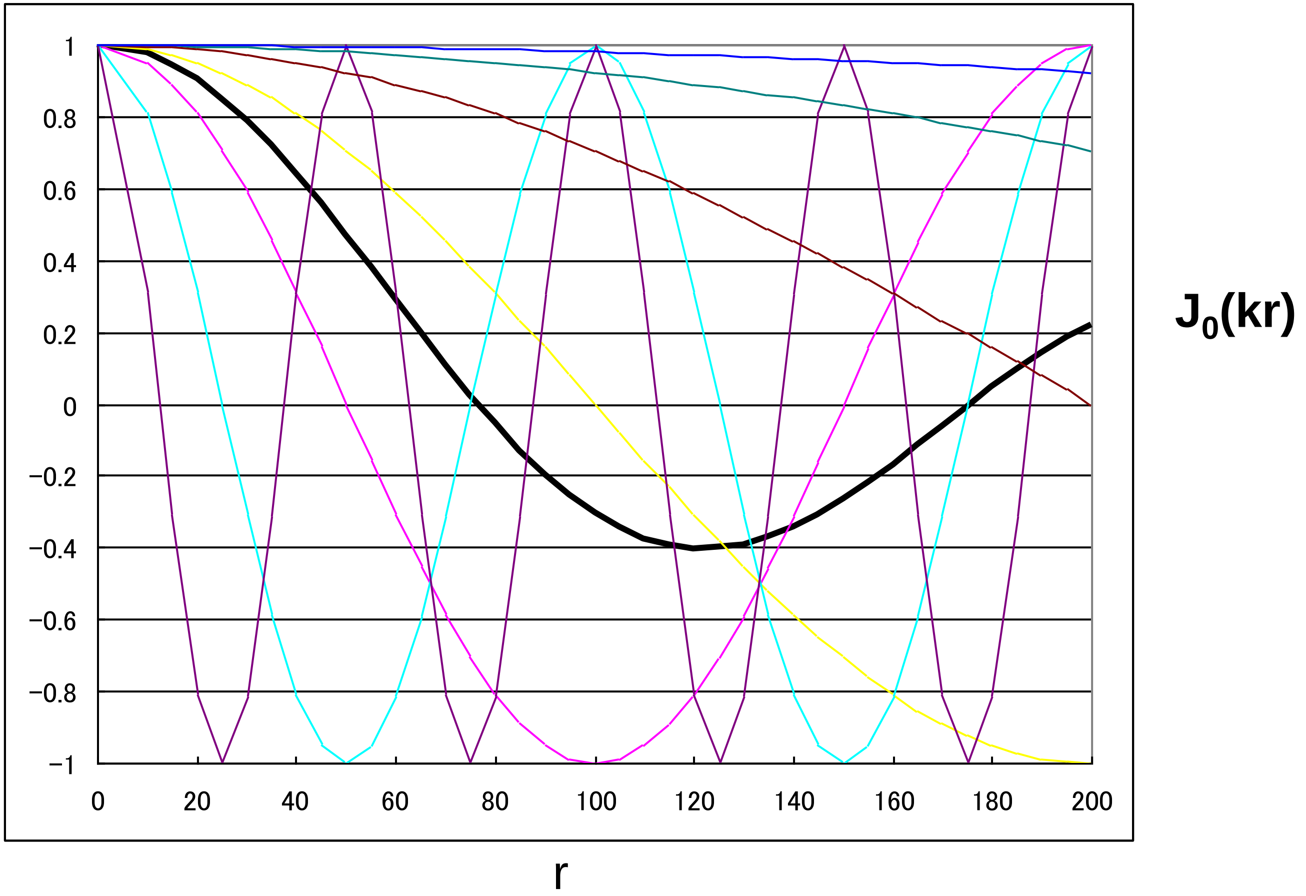
$$= \frac{1}{2\pi}\mathrm{Re}\left[\int_0^{2\pi}\exp\left(\frac{i\omega r\cos\psi}{c(\omega)}\right)d\psi\right]$$

$$= J_0\left(\frac{\omega r}{c(\omega)}\right)$$

A Mathematical Formula

$$\int_0^{2\pi}\exp(ikr\cos(\theta-\psi))\,d\theta = 2\pi J_0(kr)$$

Spatial Auto-Correlation (2D)



Aki(1957) gave the formulation for the vertical component that corresponds to Rayleigh waves.

Aki(1965) showed the extension of the theory to the horizontal components that are superposition of Rayleigh waves and Love waves.

For the horizontal component parallel to the direction among two sensors,

$$\rho_r(r,\omega) = J_0\left(\frac{r\omega}{c(\omega)}\right) - J_2\left(\frac{r\omega}{c(\omega)}\right)$$

For the horizontal component perpendicular to the direction among two sensors

$$\rho_\theta(r,\omega) = J_0\left(\frac{r\omega}{c(\omega)}\right) + J_2\left(\frac{r\omega}{c(\omega)}\right)$$

Derivation given by Aki (1957) is not easily understandable. It is recommendable to read Okada (2003, 2006) for theoretical back ground.

Assumptions used:

+Microtremor is both spatially and temporally a stationary ergodic process at and around the area where array is deployed.

+Surface waves are dominant in microtremor.

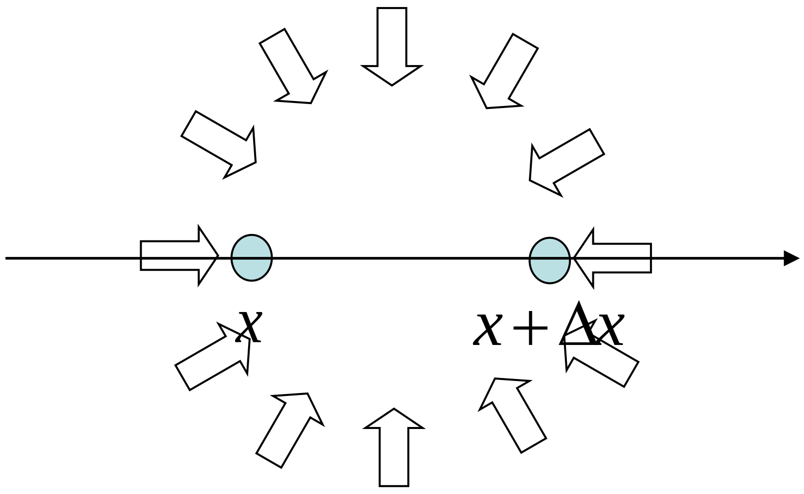
+Dominance of Single (Fundamental) mode.

+Plane waves do not interfere each other (zero correlation).

+Horizontally stratified media that is implied by propagation of plane wave with a constant velocity.

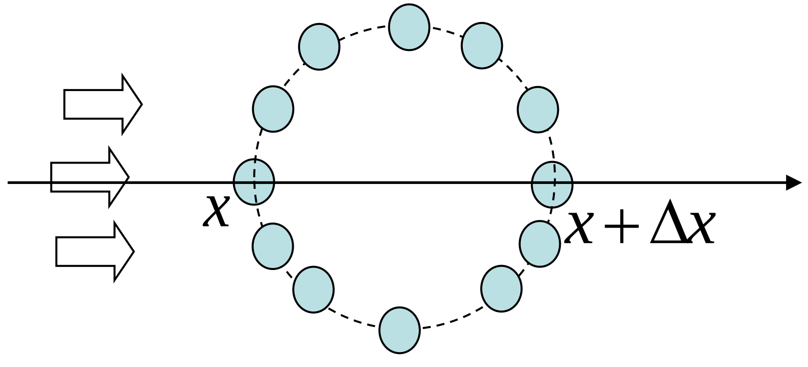
They are not always fulfilled. A possible cause of disturbance is a strong localized and temporal vibration source such as traffic near by array.

How to realize the average over propagation direction θ.



Isotropic wave field: waves come from all directions with uniform power.

Averaging can be done by wave field itself.



Anisotropic wave field: isotropic sensor arrangement of array observation can perform averaging.

Calculation of SPAC coefficient from observed data (Okada 2003)

$$\rho(r,\omega)=\frac{1}{2\pi}\int_0^{2\pi}\frac{E\left[\mathrm{Re}\left\{Cc\left((0,0),(r,\psi),\omega\right)\right\}\right]}{\sqrt{E\left[Cc\left((0,0),(0,0),\omega\right)\right]\cdot E\left[Cc\left((r,\psi),(r,\psi),\omega\right)\right]}}d\psi$$

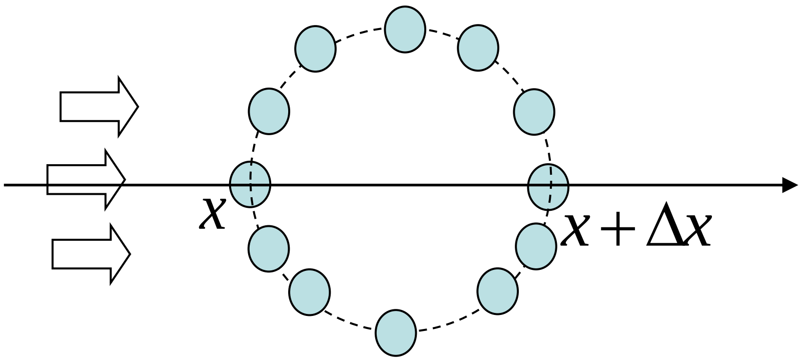
$$=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left\{Coh\left((0,0),(r,\psi),\omega\right)\right\}d\psi$$

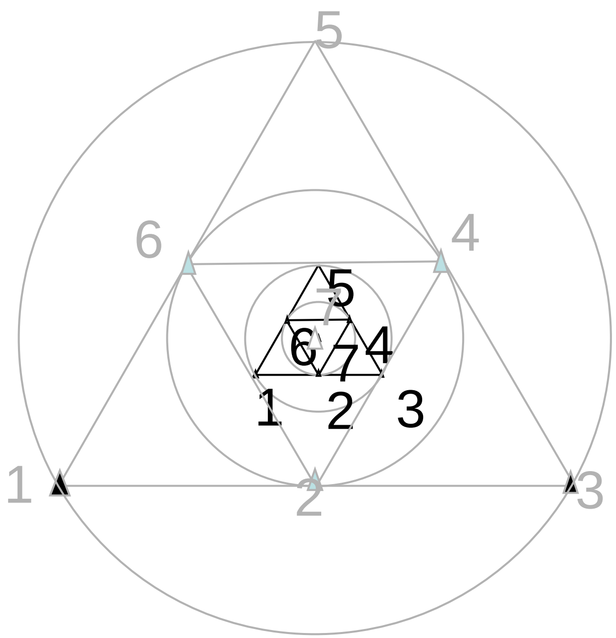
$$\approx\frac{1}{M}\sum_{m=1}^{M}\mathrm{Re}\left\{Coh\left((0,0),(r,\psi),\omega\right)\right\}$$

where *E*[ ] denotes ensamble average over time that is in practice replaced with average over time blocks. The auto-correlations in the denominator work to compensate very local amplification of microtremor.

Note that in calculation Cross-correlations are always handled by station pairs.

The way of averaging over azimuth





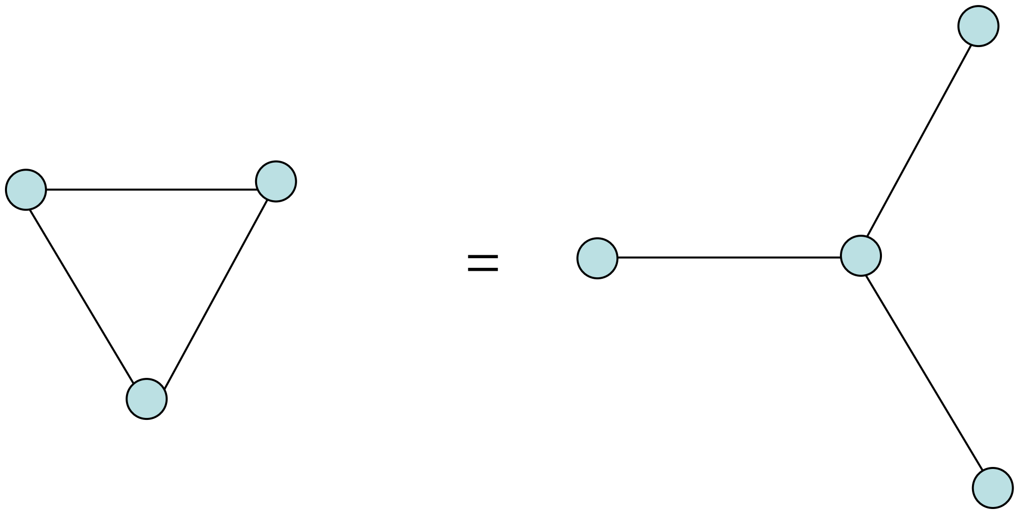
The information about the sources of microtremor is always unknown. The dependency of its power on propagating direction neither.

Bigger number of seismometer along the circle may give better averaging, however the cost is higher.

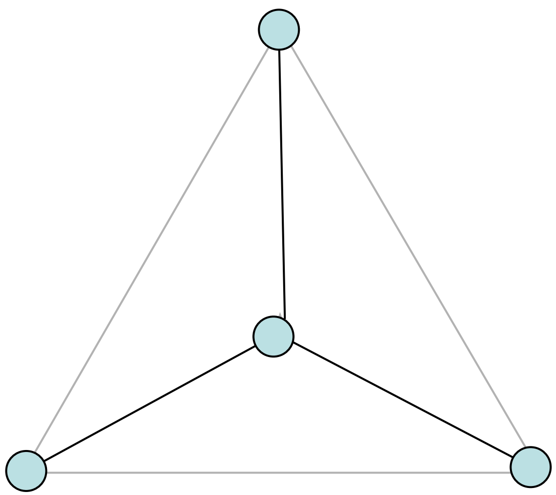
Equilateral triangle array that has three seismometers on a circle is the most efficient (Okada 2003, 2006) . However not the best.

How Many Stations / How Less Stations ?

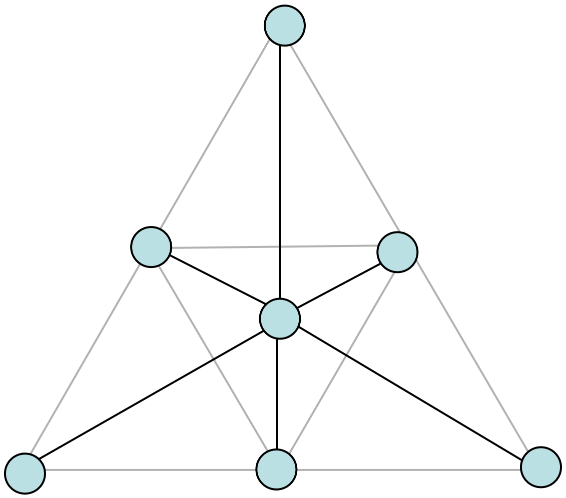
If microtremor field is stationary over space around the target area:



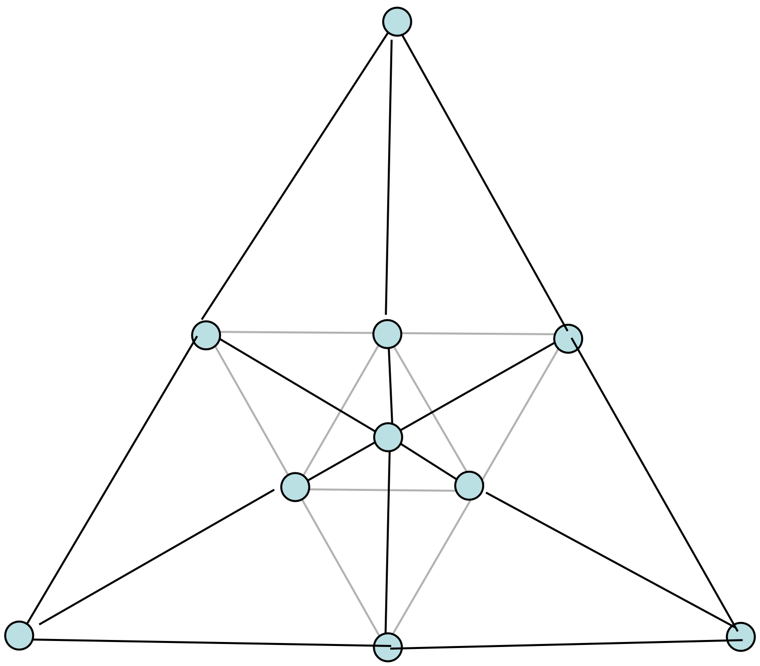
3 station array:
1 measurement gives
1 inter-station distances



4 station array:
1 measurement gives
2 inter-station distances



7 station array:
1 measurement gives
5 inter-station distances



10 station array:
1 measurement gives
8 inter-station distances

New Interpretation of SPAC Method based on the Sensitivity Analysis of array (Shiraishi et al. 2006)

Shiraishi et al. (2006) formulate the Complex Coherence Function.

$$\mathrm{Re}\left(\gamma_{pq}\right) = J_0\left(kr\right) + 2\sum_{n=1}^{\infty}\left[\left(-1\right)^n J_{2n}\left(kr\right) \times \left\{\sum_{l=1}^{L}\lambda_{pql}\cos\left(2n\theta_l\right)\right\}\right]$$

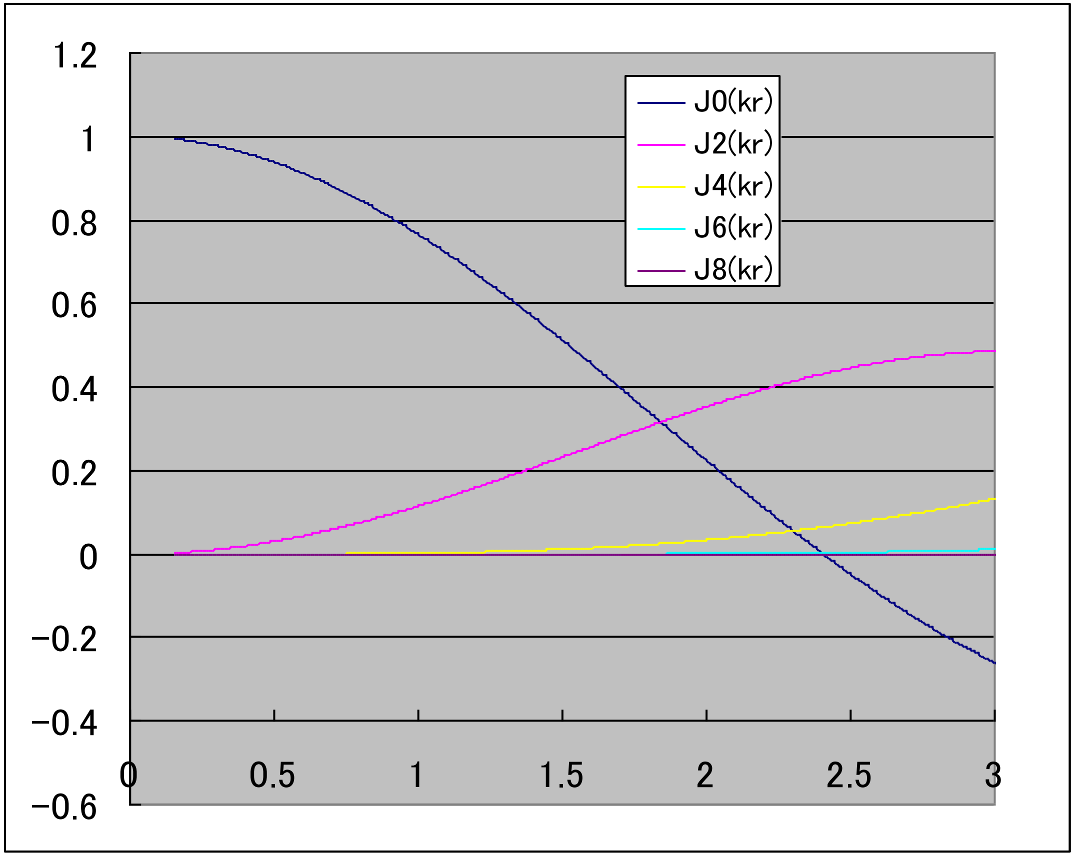
***p,q*: Number of observation points.**
***l* : Number of source**
θ_l: Azimuth of source measured from the segment connecting observation points (*p, q*)

$$\lambda_{pql} = \frac{C_{pl}C_{ql}}{\sqrt{\sum_{l=1}^{L}C_{pl}^2}\sqrt{\sum_{l=1}^{L}C_{ql}^2}}$$

The same assumptions are used as Aki(1957)

$$C_{pl} = -\sqrt{2\pi}\, f r_{pl}^{-1/2} AF_l k^{-1/2} \exp\left(-hkr_{pl}\right)$$

Bessel Function's of even order



C_{pl}: Amplitude of predominant mode of Rayleigh wave excited by *l* source and observed at *p* observation point.
***A*: Ground Response**
***h*: Damping constant of ground**
F_l: Amplitude of excitation force

For source located at enough far, the power spectra at *p*-th observation point is approximately same as that at *q*-th observation point. Thus,

$$\lambda_{pql} \approx \frac{C_{pl}^2}{\sum_{i=1}^{L} C_{pi}^2}$$

: Contribution of *l*-th source to power spectra at *p*-th or *q*-th observation points

For a line array (#1) fixed to the ground,

$$\mathrm{Re}(\gamma_{pq}) \approx J_0(k(f)r) - J_2(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(2\theta_l)\right\} + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(4\theta_l)\right\}$$

For another line array (#2) with angle ϕ from the above line array,

$$\mathrm{Re}(\gamma_{pq}) \approx J_0(k(f)r) - J_2(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(2(\theta_l + \phi))\right\} + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(4(\theta_l + \phi))\right\}$$

Sensitivity of Conventional Concentric & equi-lateral triangular array

In the conventional concentric & equi-lateral triangle array, $\phi = \pm\pi/3$ is employed, the formula corresponding three sides are the following.

$$\mathrm{Re}(\gamma_{01}) \approx J_0(k(f)r) - J_2(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{01l} \cos(2\theta_l)\right\} + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{01l} \cos(4\theta_l)\right\}$$

$$\mathrm{Re}(\gamma_{02}) \approx J_0(k(f)r) - J_2(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{02l} \cos 2\left(\theta_l - \frac{2\pi}{3}\right)\right\} + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{02l} \cos 4\left(\theta_l - \frac{2\pi}{3}\right)\right\}$$

$$\mathrm{Re}(\gamma_{03}) \approx J_0(k(f)r) - J_2(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{03l} \cos 2\left(\theta_l + \frac{2\pi}{3}\right)\right\} + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{03l} \cos 4\left(\theta_l + \frac{2\pi}{3}\right)\right\}$$

Sensitivity of Conventional Concentric & equi-lateral triangular array

The average over azimuth gives

$$\frac{\mathrm{Re}(\gamma_{01})+\mathrm{Re}(\gamma_{02})+\mathrm{Re}(\gamma_{03})}{3}\approx J_0(k(f)r)-J_2(k(f)r)\times\frac{2}{3}\left\{\sum_{l=1}^{L}\lambda_l\cos(2\theta_l)\right\}\times\left\{1+2\cos\left(\frac{4}{3}\pi\right)\right\}$$

$$+J_4(k(f)r)\times\frac{2}{3}\left\{\sum_{l=1}^{L}\lambda_l\cos(4\theta_l)\right\}\times\left\{1+2\cos\left(\frac{8}{3}\pi\right)\right\}$$

,where $\lambda_l=\lambda_{pql}=\lambda_{qpl}$, because of the spatial stationarity with in the array.

$$\left\{1+2\cos\left(\frac{4}{3}\pi\right)\right\}=1+2\times(-\frac{1}{2})=0,\ \left\{1+2\cos\left(\frac{8}{3}\pi\right)\right\}=1+2\times(-\frac{1}{2})=0$$

The terms of *$J_2(kr)$* and *$J_4(kr)$* are canceled out by the average over azimuth for any azimuth dependency of sources. Then, the conventional concentric & equi-lateral triangle array gives a simple form.

$$\rho(r,\omega)=\frac{\mathrm{Re}(\gamma_{01})+\mathrm{Re}(\gamma_{02})+\mathrm{Re}(\gamma_{03})}{3}\approx J_0(k(f)r)+o\{J_6(k(f)r)\}$$

Is L or T shape array possible?

In the conventional L or T shape array, ϕ=0, π/2 is employed, the formula corresponding two sides are the following.

$$\mathrm{Re}(\gamma_{pq}) \approx J_0(k(f)r) - J_2(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(2\theta_l)\right\} + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(4\theta_l)\right\}$$

$$\mathrm{Re}(\gamma_{pq}) \approx J_0(k(f)r) + J_2(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(2\theta_l)\right\} + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(4\theta_l)\right\}$$

The average of #1 and #2 gives

$$\mathrm{Re}(\gamma_{pq}) \approx J_0(k(f)r) + J_4(k(f)r) \times 2\left\{\sum_{l=1}^{L} \lambda_{pql} \cos(4\theta_l)\right\} + o(J_6(k(f)r))$$

This implies that L- or +- shape array is also available, if the range of $k(f)r$ is selected carefully.
For $k(f)r<2/3\pi$, the contribution of 4-th order term is negligible in usual case.

$$\mathrm{Re}(\gamma_{pq}) \approx J_0(k(f)r)$$

New Interpretation of SPAC Method based on the Seismic Interferometry (Yokoi & Margaryan 2008)

"Seismic Interferometry" refers to

Principle of generating new seismic responses by cross correlating seismic observations at different receiver locations (Wapenaar & Fokkema 2006).

In moderately azimuth dependent wave field:

$$2\,\mathrm{Re}\left\{\hat{G}^{v,\tau}_{p,q}\left(x_A, x_B, \omega\right)\right\}\hat{S}(\omega) \approx \frac{2}{\hat{\rho} c_P}\left\langle \left\{\hat{v}^{obs}_p\left(x_A, \omega\right)\right\}^* \cdot \hat{v}^{obs}_q\left(x_B, \omega\right)\right\rangle$$

p-th component of the particle velocity observed at $\boldsymbol{x_A}$ due to a unit force applied to *q*-th direction at $\boldsymbol{x_B}$

Common source power spectra

$$Cc\left(x_A, x_B, \omega\right)$$

Spatial ensamble average
(ergodicity)
temporal ensamble average

Complex coherence function

$$(\gamma_z)_{A,B} = \frac{\left\langle \left\{\hat{v}_z^{obs}(x_A,\omega)\right\}^* \hat{v}_z^{obs}(x_B\omega)\right\rangle}{\left\langle \left\{\hat{v}_z^{obs}(x_A,\omega)\right\}^* \hat{v}_z^{obs}(x_A\omega)\right\rangle} \approx \frac{-2\,\mathrm{Re}\left\{\hat{G}_{z,z}^{v,\tau}(x_A,x_B,\omega)\right\}\hat{S}_v(\omega)}{-2\,\mathrm{Re}\left\{\hat{G}_{z,z}^{v,\tau}(x_A,x_A,\omega)\right\}\hat{S}_v(\omega)}$$

$$= \frac{\mathrm{Re}\left\{\hat{G}_{z,z}^{v,\tau}(x_A,x_B,\omega)\right\}}{\mathrm{Re}\left\{\hat{G}_{z,z}^{v,\tau}(x_A,x_A,\omega)\right\}}$$

Source term is cancelled out

Assumption: dominance of Rayleigh waves

$$\hat{G}_{z,z}^{v,\tau}(x_A,x_B,\omega) \approx -\omega\sum_{n=0}^{\infty}\left\{\hat{r}_2(k_n,0)\right\}^2 J_0(k_n r_{A,B})$$

where $\hat{r}_i(k_n,z)^2 = r_i(k_n,z)^2 / 4c_n^R(\omega)U_n^R(\omega)I_1^{R(n)}(\omega)$

Eigen function of Rayleigh waves in horizontally stratified media (Aki & Richard 2002)➔Site dependent

Phase velocity

Group velocity

$$I_1^{R(n)}(\omega) = \tfrac{1}{2}\int_0^{\infty}\hat{\rho}(z)\left\{r_1(k_n,z)^2 + r_2(k_n,z)^2\right\}dz$$

Then,

$$(\gamma_z)_{A,B} \approx \frac{\mathrm{Re}\left\{\left|\hat{r}_2(k_0\ ,0)\right|^2 J_0(k_0 r_{A,B})\right\}}{\mathrm{Re}\left\{\left[\hat{r}_2(k_0\ ,0)\right]^2 J_0(0)\right\}} = J_0(k_0 r_{A,B})$$

Site dependent amplification is cancelled out

The similar derivation can be done for the horizontal components and their SPAC coefficients can be given same as Aki(1957) and Okada(2003).

The consequence of Seismic Interferometry implies

+Complex coherence function of every station pairs has physical meaning, *i. e.*, the elastodynamic Green's function normalized by its zero-off set version,

+If dependency of wave power on azimuth is enough moderate, average over azimuth can be skipped,

and completely consistent with the basic theory of SPAC and with the formulation of Shiraishi et al (2006).

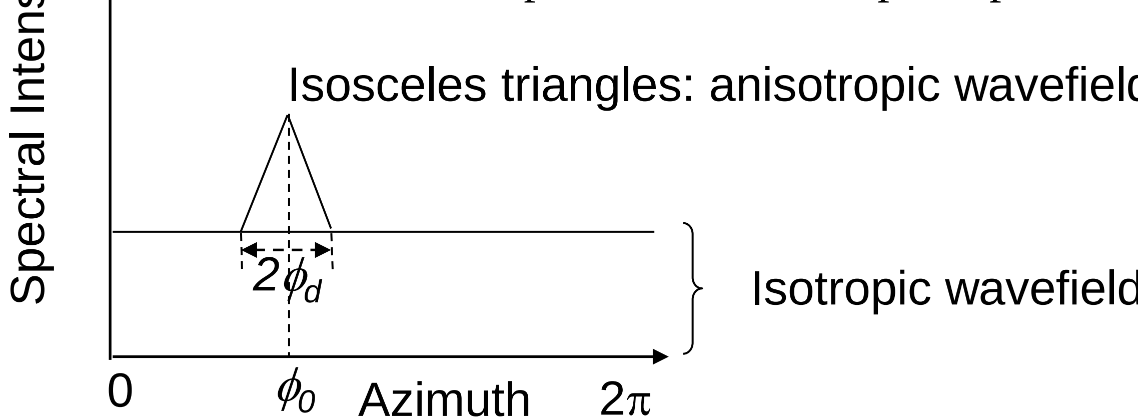
The formulation given by Aki (1957) and Okada (2003) have shown that the average over azimuth gives $\boldsymbol{J_0(kr)}$ and the complex coherence function is just an interim quantity that does not have a proper physical meaning.

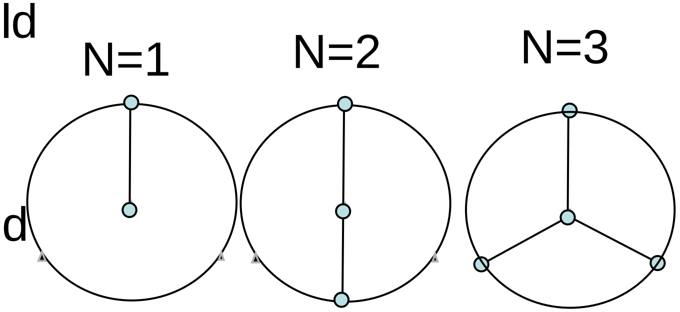
The formulation of Shiraishi et al. (2006) and the consequence of Seismic Interferometry (Yokoi and Margaryan 2008) showed that $\boldsymbol{J_0(kr)}$ occupies a major part of the complex coherence function. The average over azimuth is applied in order to cancel out the unnecessary parts that are the terms of Bessel functions of the orders higher than 2 in the formulation of Shiraishi et al. (2006). Those are the run-off from the normalized elastodynamic Green's function and its asymmetrical parts in the context of Yokoi and Margaryan (2008).

Aki(1957) showed that the average over azimuth is necessary in case of plane wave incidence from only one direction and that it can be skipped in case of isotropic wave field. Namely, only for two extreme cases. The above discussion implies that the necessity of the average over azimuth has to be considered quantitatively in relation with the required accuracy of phase velocity and the dependency of wave power on azimuth.

Cho et al.(2008) showed a quantitative assessing for this problem.

Model of microtremor field
=isotropic wave field +superimposed anisotropic wave field





Plane wave coming from one azimuth for the discrepancy of $\rho(r,\omega)$ < 0.05 times of true value
$rk<3.5$, i. e., $\lambda/r>1.8$ (N=3), $rk<0.5$, i. e., $\lambda/r>14$ (N=1 and 2)

For the discrepancy of $\rho(r,\omega)$ < 0.05 times of true value in the range $0<rk<20$
(A very severe threshold level)

Anisotropic wave field modeled by an isosceles triangle:
$\phi_d>\pi/4$ (N=3), $\phi_\delta>3\pi/4$ (N=1 and 2),

Anisotropic wave field modeled by an isosceles triangle superimposed on isotropic one where R denotes the spectral intensity ratio of isotropic one on anisotropic one.
R > 0.15 at $\phi_d=\pi/6$, R > 0.0 at $\phi_d>\pi/3$ (N=3)
R > 0.5 at $\phi_d=\pi/6$, R > 0.2 at $\phi_d=2\pi/3$, R > 0.0 at $\phi_d>7\pi/9$ (N=1)

If anisotropic wave field has enough extent over azimuth or isotropic wave field has enough contribution to total wave field, the influence of anisotropic one is negligible.

Reference List:

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